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EMBEDDINGS AND RAMSEY NUMBERS OF SPARSE k-UNIFORM HYPERGRAPHS

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Chvátal, Rödl, Szemerédi and Trotter [3] proved that the Ramsey numbers of graphs of bounded maximum degree are linear in their order. In [6,23] the same result was proved for 3-uniform hypergraphs. Here we extend this result to k-uniform hypergraphs for any integer $k \geq 3$. As in the 3-uniform case, the main new tool which we prove and use is an embedding lemma for k-uniform hypergraphs of bounded maximum degree into suitable k-uniform 'quasi-random' hypergraphs.

1. Introduction

The *Ramsey number* $R(\mathcal{H})$ of a k-uniform hypergraph $\mathcal{H}$ is the smallest $N \in \mathbb{N}$ such that for every 2-colouring of the hyperedges of the complete k-uniform hypergraph on N vertices one can find a monochromatic copy of $\mathcal{H}$. For general $\mathcal{H}$, the best upper bound is due to Erdős and Rado [7]. Writing $|\mathcal{H}|$ for the number of vertices of $\mathcal{H}$, it implies that for any $k \geq 2$

$$R(\mathcal{H}) \leq 2^{2^{\cdot^{\cdot^{\cdot^{2^{c_k|\mathcal{H}|}}}}}},$$

where the number of 2's is $k-1$. In the other direction, Erdős and Hajnal (see [11]) showed that if $k \geq 3$ and $\mathcal{H}$ is a complete k-uniform hypergraph, then $R(\mathcal{H})$ is bounded below by a tower in which the number of 2's is $k-2$ and the top exponent is $c'_k|\mathcal{H}|^2$.

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For the case of graphs (i.e., when $k=2$) it is known that there are many families of graphs H for which the Ramsey number is much smaller than exponential. In particular, Burr and Erdős [2] asked for which graphs H the Ramsey number $R(H)$ is linear in the order $|H|$ of H and conjectured this to be true for graphs of bounded maximum degree. This was proved by Chvátal, Rödl, Szemerédi and Trotter [3]. Here we show that their result extends to k-uniform hypergraphs $\mathcal{H}$ of bounded maximum degree, where the *degree of a vertex* x in $\mathcal{H}$ is defined to be the number of hyperedges which contain x.

Theorem 1. *For all $\Delta, k \in \mathbb{N}$ there exists a constant $C = C(\Delta, k)$ such that all k-uniform hypergraphs $\mathcal{H}$ of maximum degree at most Δ satisfy $R(\mathcal{H}) \le C|\mathcal{H}|$.*

The overall strategy of our proof of Theorem 1 is related to that of Chvátal et al. [3], which is based on the regularity lemma for graphs. We apply a version (due to Rödl and Schacht [27]) of the regularity lemma for k-uniform hypergraphs. Roughly speaking, it guarantees a partition of an arbitrary dense k-uniform hypergraph into 'quasi-random' subhypergraphs. Our main contribution is an embedding result (Theorem 2) which guarantees the existence of a copy of a hypergraph $\mathcal{H}$ of bounded maximum degree inside a suitable 'quasi-random' hypergraph $\mathcal{G}$ even if the order of $\mathcal{H}$ is linear in that of $\mathcal{G}$. In fact, we prove a stronger embedding result of independent interest (Theorem 3). It even counts the number of copies of such $\mathcal{H}$ in $\mathcal{G}$ and thus generalizes the well-known hypergraph counting lemma (which only allows for bounded size $\mathcal{H}$).

After the submission of this paper, Keevash [17] extended Theorem 2 to a hypergraph blow-up lemma for embeddings of spanning subhypergraphs $\mathcal{H}$. The case of 3-uniform hypergraphs in Theorem 1 was proved recently in [6] and independently by Nagle, Olsen, Rödl and Schacht [23]. Also, Kostochka and Rödl [21] earlier proved an approximate version of Theorem 1: for all $\varepsilon, \Delta, k > 0$ there is a constant C such that $R(\mathcal{H}) \le C|\mathcal{H}|^{1+\varepsilon}$ if $\mathcal{H}$ has maximum degree at most Δ. After this manuscript was submitted, Conlon, Fox and Sudakov [4] obtained a proof of Theorem 1 which does not rely on hypergraph regularity and gives a better bound on C. Also, Ishigami [16] independently announced a proof of Theorem 1 using a similar approach to ours. Apart from these, the only previous results on the Ramsey numbers of sparse hypergraphs are on hypergraph cycles (see e.g. [13–15]).

It would be desirable to extend Theorem 1 to a larger class of hypergraphs. For instance the graph analogue of Theorem 1 is known for so-called p-arrangeable graphs [1], which include the class of all planar graphs. However, Rödl and Kostochka [21] showed that a natural hypergraph analogue of the famous Burr–Erdős conjecture on Ramsey numbers of d-degenerate graphs fails for k-uniform hypergraphs if $k \geq 3$. (A graph is d-degenerate if the maximum average degree over all its subgraphs is at most d. If a graph is p-arrangeable, then it is also d-degenerate for some d.) But it may still be possible to generalize the Burr–Erdős conjecture to hypergraphs in a different way.

This paper is organized as follows. In Section 2 we give an overview of the proof of Theorem 1 and we state the embedding theorem (Theorem 2) mentioned above. Our proof of Theorem 2 relies on a more general version (Lemma 4) of the well-known counting lemma for hypergraphs as well as an 'extension lemma' (Lemma 5), whose proofs are postponed until Sections 7 and 8. We introduce these lemmas, along with further tools, in Section 3. We then prove a strengthened version (Theorem 3) of Theorem 2 in Section 4. The regularity lemma for k-uniform hypergraphs is introduced in Section 5. In Section 6 we deduce Theorem 1 from the regularity lemma and Theorem 2. In Section 7 we derive our version of the counting lemma (Lemma 4) from that in [28]. Finally, in Section 8 we use it to deduce the extension lemma (Lemma 5).

2. Overview of the proof of Theorem 1 and statement of the embedding theorem

2.1. Overview of the proof of Theorem 1

The proof in [3] that graphs of bounded degree have linear Ramsey numbers proceeds roughly as follows: Let H be a graph of maximum degree Δ. Take a complete graph K_n, where n is a sufficiently large integer. Colour the edges of K_n with red and blue, and apply the graph regularity lemma to the denser of the two monochromatic graphs, G_{red} say, to obtain a partition of the vertex set into a bounded number of clusters. Since almost all pairs of clusters are regular or 'quasi-random', by Turán's theorem there will be a set of r clusters, where $r := R(K_{\Delta+1})$, in which each pair of clusters is regular. A pair of clusters will be coloured red if its density in G_{red} is at least $1/2$, and blue otherwise. By the definition of r, there must be a set of $\Delta+1$ clusters such that all the pairs have the same colour. If this colour is red, then one can apply the so-called embedding or key lemma for graphs to find a (red) copy of H in the subgraph of G_{red} spanned by these $\Delta+1$ clusters. This is possible since $\chi(H) \leq \Delta+1$. If all the pairs of clusters are coloured blue we apply the embedding theorem in the blue subgraph G_{blue} of K_n to find a blue copy of H. It turns out that in this proof we only needed $n \geq C|H|$, where C is a constant dependent only on Δ. Thus $R(H) \leq C|H|$.

We will generalize this approach to k-uniform hypergraphs. As mentioned in Section 1, the main obstacle is the proof of an embedding theorem for k-uniform hypergraphs (Theorem 2 below), which allows us to embed a k-uniform hypergraph $\mathcal{H}$ within a suitable 'quasi-random' k-uniform hypergraph $\mathcal{G}$, where the order of $\mathcal{H}$ might be linear in the order of $\mathcal{G}$. Our proof uses ideas from [6].

2.2. Notation and statement of the embedding theorem

Before we can state the embedding theorem, we first have to say what we mean by a regular or 'quasi-random' hypergraph. In the setup below, this will involve the relationship between certain i-uniform hypergraphs and $(i-1)$-uniform hypergraphs on the same vertex set. Given a hypergraph $\mathcal{G}$, we write $E(\mathcal{G})$ for the set of its hyperedges and define $e(\mathcal{G}):=|E(\mathcal{G})|$. We write $K_i^{(j)}$ for the complete j-uniform hypergraph on i vertices. Given a j-uniform hypergraph $\mathcal{G}$ and $j\le i$, we write $\mathcal{K}_i(\mathcal{G})$ for the set of i-sets of vertices of $\mathcal{G}$ which form a copy of $K_i^{(j)}$ in $\mathcal{G}$. Given an i-partite i-uniform hypergraph $\mathcal{G}_i$, and an i-partite $(i-1)$-uniform hypergraph $\mathcal{G}_{i-1}$ on the same vertex set, we define the *density of $\mathcal{G}_i$ with respect to $\mathcal{G}_{i-1}$* to be

$$d(\mathcal{G}_i|\mathcal{G}_{i-1}) := \frac{|\mathcal{K}_i(\mathcal{G}_{i-1})\cap E(\mathcal{G}_i)|}{|\mathcal{K}_i(\mathcal{G}_{i-1})|}$$

if $|\mathcal{K}_i(\mathcal{G}_{i-1})|>0$, and $d(\mathcal{G}_i|\mathcal{G}_{i-1}):=0$ otherwise. More generally, if $\mathbf{Q}:=(Q(1),Q(2),\dots,Q(r))$ is a collection of r subhypergraphs of $\mathcal{G}_{i-1}$, we define $\mathcal{K}_i(\mathbf{Q}):=\bigcup_{j=1}^r\mathcal{K}_i(Q(j))$ and

$$d(\mathcal{G}_i|\mathbf{Q}) := \frac{|\mathcal{K}_i(\mathbf{Q})\cap E(\mathcal{G}_i)|}{|\mathcal{K}_i(\mathbf{Q})|}$$

if $|\mathcal{K}_i(\mathbf{Q})|>0$, and $d(\mathcal{G}_i|\mathbf{Q}):=0$ otherwise. We sometimes write $\left|K_i^{(j)}\right|_{\mathbf{Q}}$ instead of $|\mathcal{K}_i(\mathbf{Q})|$.

We say that $\mathcal{G}_i$ is (d_i,δ,r)*-regular with respect to* $\mathcal{G}_{i-1}$ if every r-tuple $\mathbf{Q}$ with $|\mathcal{K}_i(\mathbf{Q})|>\delta|\mathcal{K}_i(\mathcal{G}_{i-1})|$ satisfies

$$d(\mathcal{G}_i|\mathbf{Q}) = d_i\pm\delta.$$

Given $\ell\ge i\ge 3$, an ℓ-partite i-uniform hypergraph $\mathcal{G}_i$ and an ℓ-partite $(i-1)$-uniform hypergraph $\mathcal{G}_{i-1}$ on the same vertex set, we say that $\mathcal{G}_i$ is (d_i,δ,r)-regular with respect to $\mathcal{G}_{i-1}$ if for every i-tuple K of vertex classes, either $\mathcal{G}_i[K]$ is (d_i,δ,r)-regular with respect to $\mathcal{G}_{i-1}[K]$ or $d(\mathcal{G}_i[K]|\mathcal{G}_{i-1}[K])=0$

(but the latter should not hold for all K). Instead of $(d_i,\delta,1)$-regularity we sometimes refer to (d_i,δ)-*regularity*.

The density of a bipartite graph G with vertex classes A and B is defined by $d(A,B):=e(A,B)/|A||B|$ and G is (d,δ)-regular if for all sets $X\subseteq A$ and $Y\subseteq B$ with $|X|\geq\delta|A|$ and $|Y|\geq\delta|B|$ we have $d(X,Y)=d\pm\delta$. We say that an ℓ-partite graph $\mathcal{G}_2$ is (d_2,δ)-*regular* if each of the $\binom{\ell}{2}$ bipartite subgraphs forming it is either (d_2,δ)-regular or has density 0 (and if for at least one of them the former holds).

Suppose that we have $\ell\geq k$ vertex classes $V_1,\dots,V_\ell$, and that for each $i=2,\dots,k$ we are given an ℓ-partite i-uniform hypergraph $\mathcal{G}_i$ with these vertex classes. Suppose also that $\mathcal{H}$ is an ℓ-partite k-uniform hypergraph with vertex classes $X_1,\dots,X_\ell$. We will aim to embed $\mathcal{H}$ into $\mathcal{G}_k$, and in particular to embed X_j into V_j for each $j=1,\dots,\ell$. So we make the following definition: We say that $(\mathcal{G}_k,\dots,\mathcal{G}_2)$ *respects the partition of* $\mathcal{H}$ if whenever $\mathcal{H}$ contains a hyperedge with vertices in $X_{j_1},\dots,X_{j_k}$, then there is a hyperedge of $\mathcal{G}_k$ with vertices in $V_{j_1},\dots,V_{j_k}$ which also forms a copy of $K_k^{(i)}$ in $\mathcal{G}_i$ for each $i=2,\dots,k-1$.

Theorem 2 (Embedding theorem for hypergraphs). *Let Δ,k,ℓ,r,n_0 be positive integers with $k\leq\ell$ and let $c,d_2,d_3,\dots,d_k,\delta,\delta_k$ be positive constants such that $1/d_i\in\mathbb{N}$,*

$$1/n_0\ll 1/r,\delta\ll\min\{\delta_k,d_2,\dots,d_{k-1}\}\leq\delta_k\ll d_k,1/\Delta,1/\ell$$

and

$$c\ll d_2,\dots,d_k,1/\Delta,1/\ell.$$

Then the following holds for all integers $n\geq n_0$. Suppose that $\mathcal{H}$ is an ℓ-partite k-uniform hypergraph of maximum degree at most Δ with vertex classes $X_1,\dots,X_\ell$ such that $|X_i|\leq cn$ for all $i=1,\dots,\ell$. Suppose that for each $i=2,\dots,k$, $\mathcal{G}_i$ is an ℓ-partite i-uniform hypergraph with vertex classes $V_1,\dots,V_\ell$, which all have size n. Suppose also that $\mathcal{G}_k$ is (d_k,δ_k,r)-regular with respect to $\mathcal{G}_{k-1}$, that for each $i=3,\dots,k-1$, $\mathcal{G}_i$ is (d_i,δ)-regular with respect to $\mathcal{G}_{i-1}$, that $\mathcal{G}_2$ is (d_2,δ)-regular, and that $(\mathcal{G}_k,\dots,\mathcal{G}_2)$ respects the partition of $\mathcal{H}$. Then $\mathcal{G}_k$ contains a copy of $\mathcal{H}$.

In the statement of Theorem 2 we used the following notation (which will be used frequently later on as well). Given constants a_1,a_2,a_3, we write $a_1\ll a_2\ll a_3$ to mean that we choose these constants from right to left, and there are increasing functions f and g such that the lemma holds provided that $a_2\leq f(a_3)$ and $a_1\leq g(a_2)$. The functions f and g are determined by the calculations in the proof of Theorem 2, but for clarity of the exposition we will not determine them explicitly.

3. Further notation and tools

3.1. Embedding theorem for complexes

Instead of Theorem 2, we will prove a considerably stronger version which appears as Theorem 3 below. It allows the embedding of hypergraphs which are not necessarily uniform and gives a lower bound on the number of such embeddings. This enables us to prove the lemma by induction on $|\mathcal{H}|$. Before we can state Theorem 3, we need to make the following definitions.

A *complex* $\mathcal{H}$ on a vertex set V is a collection of subsets of V, each of size at least 2, such that if $B\in\mathcal{H}$, and if $A\subseteq B$ has size at least 2, then $A\in\mathcal{H}$. (So if we add each vertex in V as a singleton into a complex, we obtain a downset.) A *k-complex* is a complex in which no member has size greater than k. The members of size $i\geq 2$ are called the *i-edges* of $\mathcal{H}$ and the elements of V are called the *vertices* of $\mathcal{H}$. We write $E_i(\mathcal{H})$ for the set of all i-edges of $\mathcal{H}$ and set $e_i(\mathcal{H}):=|E_i(\mathcal{H})|$. We also write $|\mathcal{H}|:=|V|$ for the *order* of $\mathcal{H}$. Note that a k-uniform hypergraph can be turned into a k-complex by making every hyperedge into a complete i-uniform hypergraph $K_k^{(i)}$, for each $2\leq i\leq k$. (In a more general k-complex we may have i-edges which do not lie within an $(i+1)$-edge.) Given $k\leq \ell$, a *(k,ℓ)-complex* is an ℓ-partite k-complex. Given a k-complex $\mathcal{H}$, for each $i=2,\dots,k$ we denote by $\mathcal{H}_i$ the *underlying i-uniform hypergraph of* $\mathcal{H}$. So the vertices of $\mathcal{H}_i$ are those of $\mathcal{H}$ and the hyperedges of $\mathcal{H}_i$ are the i-edges of $\mathcal{H}$.

Two vertices x and y in a k-complex are *neighbours* if they are joined by a 2-edge. (Note that if x and y lie in a common i-edge for some $2\leq i\leq k$, then they are joined by a 2-edge.) The *degree* $d(x)$ of a vertex x is the maximum (over $2\leq i\leq k$) of the number of i-edges containing x. Thus x has at most $d(x)$ neighbours. The *maximum degree* of the complex $\mathcal{H}$ is the greatest degree of any vertex. Note that if $\mathcal{H}$ is a k-uniform hypergraph of maximum degree Δ, the maximum degree of the corresponding k-complex is crudely at most $\Delta 2^k$. The *distance* between two vertices x and y in a k-complex $\mathcal{H}$ is the length of the shortest path between x and y in the underlying 2-graph $\mathcal{H}_2$ of $\mathcal{H}$. The *components* of $\mathcal{H}$ are the subcomplexes induced by the components of $\mathcal{H}_2$.

We say that a k-complex $\mathcal{G}$ is $(d_k,\dots,d_2,\delta_k,\delta,r)$-regular if $\mathcal{G}_k$ is (d_k,δ_k,r)-regular with respect to $\mathcal{G}_{k-1}$, if $\mathcal{G}_i$ is (d_i,δ)-regular with respect to $\mathcal{G}_{i-1}$ for each $i=3,\dots,k-1$, and if $\mathcal{G}_2$ is (d_2,δ)-regular. We denote $(d_k,\dots,d_2)$ by $\mathbf{d}$ and refer to $(\mathbf{d},\delta_k,\delta,r)$-regularity.

Suppose that $\mathcal{G}$ is a (k,ℓ)-complex with vertex classes $V_1,\dots,V_\ell$, which all have size n. Suppose also that $\mathcal{H}$ is a (k,ℓ)-complex with vertex classes $X_1,\dots,X_\ell$ of size at most n. Similarly as for hypergraphs we say that $\mathcal{G}$

respects the partition of $\mathcal{H}$ if whenever $\mathcal{H}$ contains an i-edge with vertices in $X_{j_1},\dots,X_{j_i}$, then there is an i-edge of $\mathcal{G}$ with vertices in $V_{j_1},\dots,V_{j_i}$. On the other hand, we say that a labelled copy of $\mathcal{H}$ in $\mathcal{G}$ is *partition-respecting* if for each $i=1,\dots,\ell$ the vertices corresponding to those in X_i lie within V_i. We denote by $|\mathcal{H}|_{\mathcal{G}}$ the number of labelled, partition-respecting copies of $\mathcal{H}$ in $\mathcal{G}$.

Theorem 3 **(Embedding theorem for complexes).** *Let Δ,k,ℓ,r,n_0 be positive integers and let $c,\alpha,d_2,\dots,d_k,\delta,\delta_k$ be positive constants such that $1/d_i\in\mathbb{N}$,*

$$1/n_0 \ll 1/r,\delta \ll \min\{\delta_k,d_2,\dots,d_{k-1}\} \le \delta_k \ll \alpha \ll d_k, 1/\Delta, 1/\ell$$

and

$$c \ll \alpha, d_2,\dots,d_k.$$

Then the following holds for all integers $n\ge n_0$. Suppose that $\mathcal{H}$ is a (k,ℓ)-complex of maximum degree at most Δ with vertex classes $X_1,\dots,X_\ell$ such that $|X_i|\le cn$ for all $i=1,\dots,\ell$. Suppose also that $\mathcal{G}$ is a $(\mathbf{d},\delta_k,\delta,r)$-regular (k,ℓ)-complex with vertex classes $V_1,\dots,V_\ell$, all of size n, which respects the partition of $\mathcal{H}$. Then for every vertex h of $\mathcal{H}$ we have that

$$|\mathcal{H}|_{\mathcal{G}} \ge (1-\alpha)n\left(\prod_{i=2}^{k} d_i^{e_i(\mathcal{H})-e_i(\mathcal{H}_h)}\right)|\mathcal{H}_h|_{\mathcal{G}},$$

where $\mathcal{H}_h$ denotes the induced subcomplex of $\mathcal{H}$ obtained by removing h. In particular, $\mathcal{G}$ contains at least $((1-\alpha)n)^{|\mathcal{H}|}\prod_{i=2}^{k} d_i^{e_i(\mathcal{H})}$ labelled partition-respecting copies of $\mathcal{H}$.

As discussed in the next subsection, Theorem 3 is a generalization of the hypergraph counting lemma (which counts subcomplexes $\mathcal{H}$ of bounded order) to subcomplexes $\mathcal{H}$ of bounded degree and linear order. Note that the bound relating $|\mathcal{H}|_{\mathcal{G}}$ to $|\mathcal{H}_h|_{\mathcal{G}}$ in Theorem 3 is close to what one would get with high probability if $\mathcal{G}$ were a random complex[1]. This also shows that the bound is close to best possible. Theorem 3 will be proved in Section 4. In the proof we will need two lemmas on embeddings of complexes of bounded order, which are stated in the next subsection.

Recall that if the maximum degree of a k-uniform hypergraph $\mathcal{H}$ is at most Δ then the maximum degree of the corresponding k-complex is at most $\Delta 2^k$. So it is easy to see that Theorem 3 does indeed imply Theorem 2.

[1] That is, $\mathcal{G}_2$ is an ℓ-partite random graph with density d_2, each triangle of $\mathcal{G}_2$ is an edge of $\mathcal{G}_3$ with probability d_3 etc.

3.2. Counting lemma and extension lemma

We will need a variant (Lemma 4) of the counting lemma for k-unifom hypergraphs due to Rödl and Schacht [28, Thm. 9]. (A similar result was proved earlier by Gowers [9] as well as Nagle, Rödl and Schacht [25].) It states that if $|\mathcal{H}|$ is bounded and $\mathcal{G}$ is suitably regular, then the number of copies of $\mathcal{H}$ in $\mathcal{G}$ is as large as one would expect if $\mathcal{G}$ were random. The main difference to the result in [28] is that Lemma 4 counts copies of k-complexes $\mathcal{H}$ instead of copies of k-uniform hypergraphs $\mathcal{H}$ and also includes an upper bound on the number of these copies. We will derive Lemma 4 from the result in [28] in Section 7.

Lemma 4 (Counting lemma). *Let k,ℓ,r,t,n_0 be positive integers and let $\varepsilon,d_2,\dots,d_k,\delta,\delta_k$ be positive constants such that $1/d_i\in\mathbb{N}$ and*

$$1/n_0 \ll 1/r,\delta \ll \min\{\delta_k,d_2,\dots,d_{k-1}\} \le \delta_k \ll \varepsilon,d_k,1/\ell,1/t.$$

Then the following holds for all integers $n\ge n_0$. Suppose that $\mathcal{H}$ is a (k,ℓ)-complex on t vertices with vertex classes $X_1,\dots,X_\ell$. Suppose also that $\mathcal{G}$ is a $(\mathbf{d},\delta_k,\delta,r)$-regular (k,ℓ)-complex with vertex classes $V_1,\dots,V_\ell$, all of size n, which respects the partition of $\mathcal{H}$. Then

$$|\mathcal{H}|_{\mathcal{G}} = (1\pm\varepsilon)n^t\prod_{i=2}^{k} d_i^{e_i(\mathcal{H})}.$$

The main difference between the counting lemma and Theorem 3 is that the counting lemma only allows for complexes $\mathcal{H}$ of bounded order. We will apply the counting lemma to embed complexes of order at most $f(\varDelta,k)$ for some appropriate function f. Note that the upper and lower bounds of the counting lemma imply Theorem 3 for the case when $|\mathcal{H}|$ is bounded. A formal proof of this (which settles the base case for the induction in the proof of Theorem 3) can be found at the beginning of Section 4.

In the induction step of the proof of Theorem 3 we will also need the following extension lemma, which states that if $\mathcal{H}'$ is a complex of bounded order, $\mathcal{H}\subseteq\mathcal{H}'$ is an induced subcomplex and $\mathcal{G}$ is suitably regular, then almost all copies of $\mathcal{H}$ in $\mathcal{G}$ can be extended to about the 'right' number of copies of $\mathcal{H}'$, where the 'right' number is the number one would expect if $\mathcal{G}$ were random. We will derive Lemma 5 from Lemma 4 in Section 8.

Lemma 5 (Extension lemma). *Let k,ℓ,r,t,t',n_0 be positive integers, where $t<t'$, and let $\beta,d_2,\dots,d_k,\delta,\delta_k$ be positive constants such that $1/d_i\in\mathbb{N}$ and*

$$1/n_0 \ll 1/r,\delta \ll \min\{\delta_k,d_2,\dots,d_{k-1}\} \le \delta_k \ll \beta,d_k,1/\ell,1/t'.$$

Then the following holds for all integers $n \geq n_0$. Suppose that $\mathcal{H}'$ is a (k,ℓ)-complex on t' vertices with vertex classes $X_1,\dots,X_\ell$ and let $\mathcal{H}$ be an induced subcomplex of $\mathcal{H}'$ on t vertices. Suppose also that $\mathcal{G}$ is a $(\mathbf{d},\delta_k,\delta,r)$-regular (k,ℓ)-complex with vertex classes $V_1,\dots,V_\ell$, all of size n, which respects the partition of $\mathcal{H}'$. Then all but at most $\beta|\mathcal{H}|_{\mathcal{G}}$ labelled partition-respecting copies of $\mathcal{H}$ in $\mathcal{G}$ are extendible to

$$(1\pm\beta)n^{t'-t}\prod_{i=2}^{k} d_i^{e_i(\mathcal{H}')-e_i(\mathcal{H})}$$

labelled partition-respecting copies of $\mathcal{H}'$ in $\mathcal{G}$.

As well as these versions of the counting lemma and the extension lemma, we will need to be able to apply versions of these lemmas to underlying $(k-1)$-complexes. In this case, we have that the regularity constant δ is much smaller than all the densities $d_2,\dots,d_{k-1}$, but on the other hand we have no r in the highest level and thus we cannot apply Lemmas 4 and 5. So instead of Lemma 4 we will use the following variant of a result of Kohayakawa, Rödl and Skokan [18, Cor. 6.11].

Lemma 6 (Dense counting lemma). *Let k,ℓ,t,n_0 be positive integers and let $\varepsilon,d_2,\dots,d_{k-1},\delta$ be positive constants such that*

$$1/n_0 \ll \delta \ll \varepsilon \ll d_2,\dots,d_{k-1},1/\ell,1/t.$$

Then the following holds for all integers $n \geq n_0$. Suppose that $\mathcal{H}$ is a $(k-1,\ell)$-complex on t vertices with vertex classes $X_1,\dots,X_\ell$. Suppose also that $\mathcal{G}$ is a $(d_{k-1},\dots,d_2,\delta,\delta,1)$-regular $(k-1,\ell)$-complex with vertex classes $V_1,\dots,V_\ell$, all of size n, which respects the partition of $\mathcal{H}$. Then

$$|\mathcal{H}|_{\mathcal{G}} = (1\pm\varepsilon)n^t\prod_{i=2}^{k-1} d_i^{e_i(\mathcal{H})}.$$

In Section 7 we will show how Lemma 6 can be deduced from the result in [18]. The following dense version of the extension lemma can be deduced from the dense counting lemma (see Section 8).

Lemma 7 (Dense extension lemma). *Let k,ℓ,t,t',n_0 be positive integers and let $\beta,d_2,\dots,d_{k-1},\delta$ be positive constants such that*

$$1/n_0 \ll \delta \ll \beta \ll d_2,\dots,d_{k-1},1/\ell,1/t'.$$

Then the following holds for all integers $n \geq n_0$. Suppose that $\mathcal{H}'$ is a $(k-1,\ell)$-complex on t' vertices with vertex classes $X_1,\dots,X_\ell$ and let $\mathcal{H}$ be an induced *subcomplex of $\mathcal{H}'$ on t vertices. Suppose also that $\mathcal{G}$ is a $(d_{k-1},\dots,d_2,\delta,\delta,1)$-regular $(k-1,\ell)$-complex with vertex classes $V_1,\dots,V_\ell$, all of size n, which respects the partition of $\mathcal{H}'$. Then all but at most $\beta|\mathcal{H}|_{\mathcal{G}}$ labelled partition-respecting copies of $\mathcal{H}$ in $\mathcal{G}$ can be extended into*

$$(1\pm\beta)n^{|\mathcal{H}'|-|\mathcal{H}|}\prod_{i=2}^{k-1} d_i^{e_i(\mathcal{H}')-e_i(\mathcal{H})}$$

labelled partition-respecting copies of $\mathcal{H}'$ in $\mathcal{G}$.

An overview of how all these lemmas are used in the proof of Theorem 1 is shown in Figure 1.

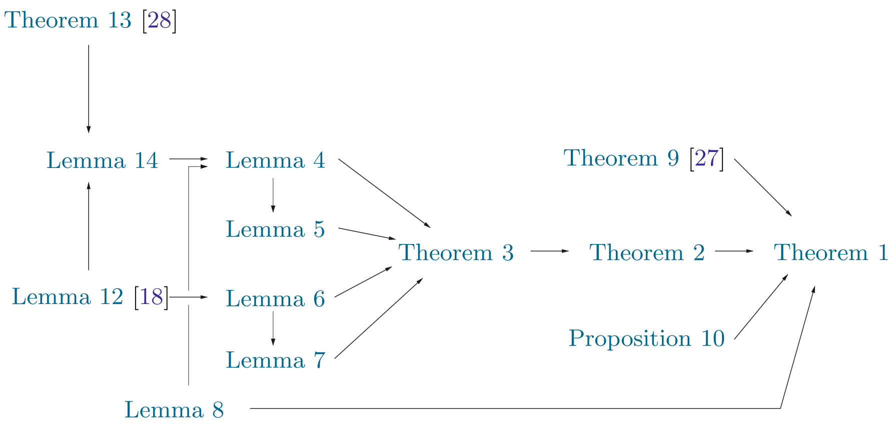

Figure 1. Proof of Theorem 1 – Flowchart

Another auxiliary result that we will use in the proof of Lemma 4 as well as in the proof of Theorem 1 is the slicing lemma. Roughly speaking, this says that in a regular complex $\mathcal{G}$, we can partition the edge set $E_j(\mathcal{G})$ of the jth level into an arbitrary number of parts so that each part is still regular with respect to $\mathcal{G}_{j-1}$ with the appropriate density, at the expense of a larger regularity constant. This can be proved using a simple application of a Chernoff bound.

Lemma 8 (Slicing lemma [27]). *Let $j\geq 2$ and $s_0,r\geq 1$ be integers and let δ_0,d_0 and p_0 be positive real numbers. Then there is an integer $n_0 = n_0(j,s_0,r,\delta_0,d_0,p_0)$ such that the following holds. Let $n\geq n_0$ and let $\mathcal{G}_j$ be a j-partite j-uniform hypergraph with vertex classes $V_1,\dots,V_j$ which all have size n. Also let $\mathcal{G}_{j-1}$ be a j-partite $(j-1)$-uniform hypergraph with the same* *vertex classes and assume that each j-set of vertices that spans a hyperedge in $\mathcal{G}_j$ also spans a $K_j^{(j-1)}$ in $\mathcal{G}_{j-1}$. Suppose that*

1. $|\mathcal{K}_j(\mathcal{G}_{j-1})| > n^j/\ln n$ *and*
2. $\mathcal{G}_j$ *is (d,δ,r)-regular with respect to $\mathcal{G}_{j-1}$, where* $d \geq d_0 \geq 2\delta \geq 2\delta_0$.

Then for any positive integer $s \leq s_0$ and all positive reals $p_1,\dots,p_s \geq p_0$ with $\sum_{i=1}^{s} p_i \leq 1$ there exists a partition of $E(\mathcal{G}_j)$ into $s+1$ parts $E^{(0)}(\mathcal{G}_j), E^{(1)}(\mathcal{G}_j),\dots,E^{(s)}(\mathcal{G}_j)$ such that if $\mathcal{G}_j(i)$ denotes the spanning subhypergraph of $\mathcal{G}_j$ whose edge set is $E^{(i)}(\mathcal{G}_j)$, then $\mathcal{G}_j(i)$ is $(p_i d, 3\delta, r)$-regular with respect to $\mathcal{G}_{j-1}$ for every $i=1,\dots,s$.

Moreover, $\mathcal{G}_j(0)$ is $((1-\sum_{i=1}^{s} p_i)\,d, 3\delta, r)$-regular with respect to $\mathcal{G}_{j-1}$ and $E^{(0)}(\mathcal{G}_j)=\emptyset$ if $\sum_{i=1}^{s} p_i = 1$.

4. Proof of the embedding theorem for complexes (Theorem 3)

Throughout the rest of the paper, whenever we talk about a copy of a complex $\mathcal{H}$ in $\mathcal{G}$ we mean that this copy is labelled and partition-respecting, without mentioning this explicitly. We prove Theorem 3 by induction on $|\mathcal{H}|$. [6] contains a sketch of the argument for the graph case which gives a good idea of the proof. We first suppose that the connected component of $\mathcal{H}$ which contains the vertex h has order less than Δ^5. In this case we will use the counting lemma to prove the embedding theorem. So let $\mathcal{C}$ be the component of $\mathcal{H}$ containing h, and let $\mathcal{D} := \mathcal{H} - \mathcal{C}$. Also, let $\mathcal{C}_h := \mathcal{C} - h$. We may assume that both $\mathcal{C}_h$ and $\mathcal{D}$ are non-empty. (If $\mathcal{D}$ is empty then the result follows from Lemma 5, and if $\mathcal{C}_h$ is empty then h is an isolated vertex and the result is trivial.) Note that a copy of $\mathcal{H}$ consists of disjoint copies of $\mathcal{C}$ and $\mathcal{D}$, while $\mathcal{H}_h$ consists of disjoint copies of $\mathcal{C}_h$ and $\mathcal{D}$. Copies of these complexes in $\mathcal{G}$ will be denoted by C, D and C_h.

Choose a new constant β such that $c, \delta_k \ll \beta \ll \alpha$. Now note that $|\mathcal{H}|_{\mathcal{G}} = \sum_{D \subseteq \mathcal{G}} |\mathcal{C}|_{\mathcal{G}-D}$, and by applying the upper and lower bounds of the counting lemma to copies of $\mathcal{C}$ in $\mathcal{G}$ and $\mathcal{G}-D$ respectively, we obtain $|\mathcal{C}|_{\mathcal{G}-D} \geq \frac{(1-c)^{\Delta^5}(1-\beta)}{(1+\beta)} |\mathcal{C}|_{\mathcal{G}} \geq (1-3\beta)|\mathcal{C}|_{\mathcal{G}}$. So

$$|\mathcal{H}|_{\mathcal{G}} \geq \sum_{D \subseteq \mathcal{G}} (1-3\beta)|\mathcal{C}|_{\mathcal{G}} = (1-3\beta)|\mathcal{C}|_{\mathcal{G}}|\mathcal{D}|_{\mathcal{G}}. \tag{1}$$

On the other hand, by a similar argument using the upper and lower bounds from the counting lemma in $\mathcal{G}$ for $\mathcal{C}_h$ and $\mathcal{C}$ respectively,

$$|\mathcal{H}_h|_{\mathcal{G}} \leq |\mathcal{C}_h|_{\mathcal{G}}|\mathcal{D}|_{\mathcal{G}} \leq \frac{1+\beta}{1-\beta} \frac{|\mathcal{C}|_{\mathcal{G}}|\mathcal{D}|_{\mathcal{G}}}{n \prod_{i=2}^{k} d_i^{e_i(\mathcal{C})-e_i(\mathcal{C}_h)}}. \tag{2}$$

Combining (1) and (2) gives the desired result.

Thus we may assume that the component of $\mathcal{H}$ containing h has order at least Δ^5. This deals with the base case of the inductive argument, and it also means that the fourth neighbourhood of h in $\mathcal{H}$ will be non-empty, which will be convenient later on in the proof as we will only be counting complexes which are non-empty.

We pick new constants ε_k and ε_{k-1} satisfying the following hierarchies:

$$\delta \ll \varepsilon_{k-1} \ll d_2, d_3, \dots, d_k, 1/\Delta;$$
$$c, \delta_k, \varepsilon_{k-1} \ll \varepsilon_k \ll \alpha.$$

Let $\mathcal{N}_h$ be the subcomplex of $\mathcal{H}$ induced by the neighbours of h, and let $\mathcal{B}$ be the subcomplex of $\mathcal{H}$ induced by h and the neighbours of h. Then any copy of $\mathcal{H}$ in $\mathcal{G}$ extending a copy N_h of $\mathcal{N}_h$ can be obtained by first extending N_h into a copy of $\mathcal{H}_h$ and then extending N_h into a copy of $\mathcal{B}$, where the vertex chosen for h has to be distinct from all the vertices chosen for $\mathcal{H}_h$.

We now introduce some more notation. Given k-complexes $\mathcal{H}' \subseteq \mathcal{H}''$ such that $\mathcal{H}'$ is induced, and a copy H' of $\mathcal{H}'$ in $\mathcal{G}$, we define $|H' \to \mathcal{H}''|$ to be the number of ways in which H' can be extended to a copy of $\mathcal{H}''$ in $\mathcal{G}$. We also define

$$\overline{|\mathcal{H}' \to \mathcal{H}''|} := n^{|\mathcal{H}''|-|\mathcal{H}'|} \prod_{i=2}^{k} d_i^{e_i(\mathcal{H}'')-e_i(\mathcal{H}')}.$$

Thus $\overline{|\mathcal{H}' \to \mathcal{H}''|}$ is roughly the expected number of ways H' could be extended to a copy of $\mathcal{H}''$ if $\mathcal{G}$ were a random complex.

We define a copy N_h of $\mathcal{N}_h$ to be *typical* if it has about the correct number of extensions into $\mathcal{B}$, i.e., if $|N_h \to \mathcal{B}| = (1 \pm \varepsilon_k)\overline{|\mathcal{N}_h \to \mathcal{B}|}$. An application of the extension lemma (Lemma 5) shows that at most $\varepsilon_k |\mathcal{N}_h|_{\mathcal{G}}$ copies of $\mathcal{N}_h$ in $\mathcal{G}$ are not typical. We denote the set of typical copies of $\mathcal{N}_h$ by `typ`, and the set of all atypical copies by `atyp`.

Now observe that if all of the copies of $\mathcal{N}_h$ were typical, the proof would be complete, since then

$$\begin{aligned} |\mathcal{H}|_{\mathcal{G}} &\geq \sum_{N_h \subseteq \mathcal{G}} |N_h \to \mathcal{H}_h|(|N_h \to \mathcal{B}| - cn) \\ &\geq \left((1-\varepsilon_k)\overline{|\mathcal{N}_h \to \mathcal{B}|} - cn\right) \sum_{N_h \subseteq \mathcal{G}} |N_h \to \mathcal{H}_h| \\ &\geq (1-\alpha)\overline{|\mathcal{N}_h \to \mathcal{B}|}|\mathcal{H}_h|_{\mathcal{G}} = (1-\alpha)\overline{|\mathcal{H}_h \to \mathcal{H}|}|\mathcal{H}_h|_{\mathcal{G}}. \end{aligned} \tag{3}$$

The third inequality follows since $c \ll \alpha, d_2, \dots, d_k$, and $\varepsilon_k \ll \alpha$.

However, we also need to take account of the atypical copies of $\mathcal{N}_h$. The proportion of these is about ε_k, which may be larger than some d_i. It will turn out that this is too large for our purposes, and so we will need to consider the atypical copies more carefully.

We define, instead of $|H' \to \mathcal{H}''|$, the expression $\left|H' \overset{k-1}{\to} \mathcal{H}''\right|$, where $\mathcal{H}' \subseteq \mathcal{H}''$ are induced subcomplexes of $\mathcal{H}$ and H' is a copy of $\mathcal{H}'$ in $\mathcal{G}$. We consider the underlying $(k-1)$-complexes in each case, and define $\left|H' \overset{k-1}{\to} \mathcal{H}''\right|$ to be the number of ways in which the underlying $(k-1)$-complex of H' can be extended to the underlying $(k-1)$-complex of $\mathcal{H}''$ within (the underlying $(k-1)$-complex of) $\mathcal{G}$. Clearly $\left|H' \overset{k-1}{\to} \mathcal{H}''\right| \geq |H' \to \mathcal{H}''|$. We also define

$$\overline{\left|\mathcal{H}' \overset{k-1}{\to} \mathcal{H}''\right|} := n^{|\mathcal{H}''|-|\mathcal{H}'|} \prod_{i=2}^{k-1} d_i^{e_i(\mathcal{H}'')-e_i(\mathcal{H}')}.$$

Thus $\overline{\left|\mathcal{H}' \overset{k-1}{\to} \mathcal{H}''\right|}$ is roughly the expected value of $\left|H' \overset{k-1}{\to} \mathcal{H}''\right|$ if $\mathcal{G}$ were a random complex. Also,

$$\overline{\left|\mathcal{H}' \overset{k-1}{\to} \mathcal{H}''\right|} = \overline{|\mathcal{H}' \to \mathcal{H}''|}/d_k^{e_k(\mathcal{H}'')-e_k(\mathcal{H}')} \geq \overline{|\mathcal{H}' \to \mathcal{H}''|}.$$

We define $\mathcal{N}_h^*$ to be the subcomplex of $\mathcal{H}$ induced by the vertices at distance 3 from h. We also define $\mathcal{F}$ to be the subcomplex of $\mathcal{H}$ induced by the vertices at distance 1, 2 or 3 from h, i.e., the subcomplex induced by $\mathcal{N}_h$, $\mathcal{N}_h^*$ and the vertices in between (see Figure 2).

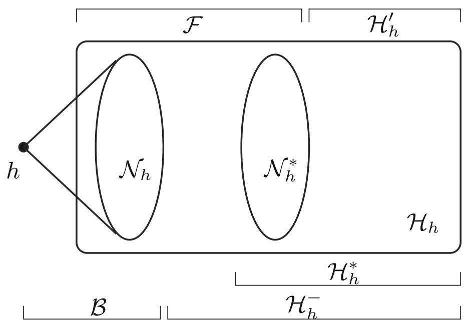

Figure 2. The complex $\mathcal{H}$

Given copies N_h of $\mathcal{N}_h$ and N_h^* of $\mathcal{N}_h^*$, we say that the pair N_h, N_h^* is *useful* if N_h and N_h^* are disjoint and if the pair has about the expected number of extensions into copies of $\mathcal{F}$ as $(k-1)$-complexes, i.e., if

$$\left|N_h \cup N_h^* \overset{k-1}{\to} \mathcal{F}\right| = (1 \pm \varepsilon_{k-1})\overline{\left|\mathcal{N}_h \cup \mathcal{N}_h^* \overset{k-1}{\to} \mathcal{F}\right|}.$$

We use Lemmas 4, 6 and 7 applied to $\mathcal{N}_h \cup \mathcal{N}_h^*$ to show that at most $\sqrt{\varepsilon_{k-1}}|\mathcal{N}_h|_{\mathcal{G}}|\mathcal{N}_h^*|_{\mathcal{G}}$ disjoint pairs N_h, N_h^* are not useful. Let $|\mathcal{N}_h \cup \mathcal{N}_h^*|_{\mathcal{G}}^{(k-1)}$ denote the number of copies of the underlying $(k-1)$-complex of $\mathcal{N}_h \cup \mathcal{N}_h^*$ in $\mathcal{G}$. Then Lemmas 4 and 6 together imply that $|\mathcal{N}_h \cup \mathcal{N}_h^*|_{\mathcal{G}}^{(k-1)} \leq (1+2\varepsilon_k) \cdot |\mathcal{N}_h \cup \mathcal{N}_h^*|_{\mathcal{G}}/d_k^{e_k(\mathcal{N}_h \cup \mathcal{N}_h^*)}$. Moreover, the dense extension lemma (Lemma 7) shows that all but at most $\varepsilon_{k-1}|\mathcal{N}_h \cup \mathcal{N}_h^*|_{\mathcal{G}}^{(k-1)}$ copies of the underlying $(k-1)$-complex of $\mathcal{N}_h \cup \mathcal{N}_h^*$ in $\mathcal{G}$ are useful. Altogether this shows that all but at most

$$\begin{aligned} \varepsilon_{k-1}(1+2\varepsilon_k)|\mathcal{N}_h \cup \mathcal{N}_h^*|_{\mathcal{G}}/d_k^{e_k(\mathcal{N}_h \cup \mathcal{N}_h^*)} &\leq \sqrt{\varepsilon_{k-1}}|\mathcal{N}_h \cup \mathcal{N}_h^*|_{\mathcal{G}} \\ &\leq \sqrt{\varepsilon_{k-1}}|\mathcal{N}_h|_{\mathcal{G}}|\mathcal{N}_h^*|_{\mathcal{G}} \end{aligned} \tag{4}$$

disjoint pairs of copies of $\mathcal{N}_h$ and $\mathcal{N}_h^*$ are useful. Note that if we had chosen $\mathcal{N}_h^*$ to be the subcomplex of $\mathcal{H}$ induced by the vertices at distance 2 from h (instead of 3), then we could not have applied Lemma 6, since $\mathcal{N}_h \cup \mathcal{N}_h^*$ would not be an induced subcomplex of $\mathcal{F}$. Together with the fact that only comparatively few of the pairs N_h, N_h^* will intersect, this shows that at most $2\sqrt{\varepsilon_{k-1}}|\mathcal{N}_h|_{\mathcal{G}}|\mathcal{N}_h^*|_{\mathcal{G}}$ pairs N_h, N_h^* are not useful. Hence at most $\varepsilon_{k-1}^{1/4}|\mathcal{N}_h|_{\mathcal{G}}$ copies of $\mathcal{N}_h$ form a non-useful pair together with more than $2\varepsilon_{k-1}^{1/4}|\mathcal{N}_h^*|_{\mathcal{G}}$ copies of $\mathcal{N}_h^*$. We call all other copies of $\mathcal{N}_h$ *useful* and let $\texttt{Usef}$ denote the set of all these copies. Then

$$|\mathcal{N}_h|_{\mathcal{G}} - |\texttt{Usef}| \leq \varepsilon_{k-1}^{1/4}|\mathcal{N}_h|_{\mathcal{G}}. \tag{5}$$

We denote by $\texttt{Usef}^*(N_h)$ the set of all N_h^* which form a useful pair together with N_h.

Claim. *Any useful copy N_h of $\mathcal{N}_h$ satisfies*

$$|N_h \to \mathcal{H}_h| \leq \frac{10}{d_k^{\Delta^3}} \frac{|\mathcal{H}_h|_{\mathcal{G}}}{|\mathcal{N}_h|_{\mathcal{G}}}.$$

Note that $\sum_{N_h} |N_h \to \mathcal{H}_h| = |\mathcal{H}_h|_{\mathcal{G}}$, so $|\mathcal{H}_h|_{\mathcal{G}}/|\mathcal{N}_h|_{\mathcal{G}}$ is the average value of $|N_h \to \mathcal{H}_h|$ over all copies N_h of $\mathcal{N}_h$. Later on, we will apply the claim to show that only a small fraction of copies of $\mathcal{H}$ contain a useful but atypical copy of $\mathcal{N}_h$.

Proof of Claim. Fix a useful copy N_h of $\mathcal{N}_h$. Put $\mathcal{H}_h^* := \mathcal{H}_h - (\mathcal{F} - \mathcal{N}_h^*)$. We aim to extend N_h to a copy of $\mathcal{H}_h$ by first picking a copy N_h^* of $\mathcal{N}_h^*$, then extending this to a copy of $\mathcal{H}_h^*$ and also extending $N_h \cup N_h^*$ to a copy of $\mathcal{F}$. We must also make sure that no vertices are used more than once. However, since we are only looking for an upper bound on $|N_h \to \mathcal{H}_h|$, and ignoring this restriction can only increase the number of extensions we find, we may ignore this difficulty. Thus

$$|N_h \to \mathcal{H}_h| \leq \sum_{N_h^* \in \mathtt{Usef}^*(N_h)} |N_h \cup N_h^* \to \mathcal{F}||N_h^* \to \mathcal{H}_h^*| + \sum_{N_h^* \notin \mathtt{Usef}^*(N_h)} |N_h \cup N_h^* \to \mathcal{F}||N_h^* \to \mathcal{H}_h^*|. \tag{6}$$

We bound the two sums separately. To bound the first sum, we need to bound $|N_h \cup N_h^* \to \mathcal{F}|$ in the case when the pair N_h, N_h^* is useful. But clearly $|N_h \cup N_h^* \to \mathcal{F}| \leq \left|N_h \cup N_h^* \overset{k-1}{\to} \mathcal{F}\right|$, and

$$\left|N_h \cup N_h^* \overset{k-1}{\to} \mathcal{F}\right| \leq (1+\varepsilon_{k-1})\overline{\left|\mathcal{N}_h \cup \mathcal{N}_h^* \overset{k-1}{\to} \mathcal{F}\right|} = \frac{(1+\varepsilon_{k-1})\overline{|\mathcal{N}_h \cup \mathcal{N}_h^* \to \mathcal{F}|}}{d_k^{e_k(\mathcal{F}) - e_k(\mathcal{N}_h) - e_k(\mathcal{N}_h^*)}}$$

whenever $N_h^* \in \mathtt{Usef}^*(N_h)$. So the first sum in (6) is bounded by

$$\frac{1+\varepsilon_{k-1}}{d_k^{e_k(\mathcal{F}) - e_k(\mathcal{N}_h) - e_k(\mathcal{N}_h^*)}} \overline{|\mathcal{N}_h \cup \mathcal{N}_h^* \to \mathcal{F}|} |\mathcal{H}_h^*|_{\mathcal{G}} \leq \frac{2}{d_k^{\Delta^3}} \overline{|\mathcal{N}_h \cup \mathcal{N}_h^* \to \mathcal{F}|} |\mathcal{H}_h^*|_{\mathcal{G}}. \tag{7}$$

To see the bound of Δ^3 on the number of k-edges which we used in the final inequality, note that $|\mathcal{F} - \mathcal{N}_h - \mathcal{N}_h^*| \leq \Delta^2$ and that the number of k-edges each of these vertices lies in is at most Δ. We now want to express the bound in (7) in terms of $|\mathcal{H}_h^-|_{\mathcal{G}}$, where $\mathcal{H}_h^- := \mathcal{H}_h - \mathcal{N}_h$. By the induction hypothesis applied several times,

$$\begin{aligned} |\mathcal{H}_h^*|_{\mathcal{G}} &\leq ((1-\alpha)n)^{-(|\mathcal{H}_h^-| - |\mathcal{H}_h^*|)} \left(\prod_{i=2}^{k} d_i^{-(e_i(\mathcal{H}_h^-) - e_i(\mathcal{H}_h^*))} \right) |\mathcal{H}_h^-|_{\mathcal{G}} \\ &\leq 2 \frac{\prod_{i=2}^{k} d_i^{e_i(\mathcal{H}_h) - e_i(\mathcal{H}_h^-) - e_i(\mathcal{N}_h)}}{\overline{|\mathcal{N}_h \cup \mathcal{N}_h^* \to \mathcal{F}|}} |\mathcal{H}_h^-|_{\mathcal{G}}. \end{aligned}$$

In the last line we used that $e_i(\mathcal{H}_h) = e_i(\mathcal{H}_h^*) + e_i(\mathcal{F}) - e_i(\mathcal{N}_h^*)$ and $|\mathcal{F}| - |\mathcal{N}_h| - |\mathcal{N}_h^*| = |\mathcal{H}_h^-| - |\mathcal{H}_h^*|$ (see Figure 2). We also used that $(1-\alpha)^{-(|\mathcal{H}_h^-| - |\mathcal{H}_h^*|)} \leq 2$.

So we obtain

$$\sum_{N_h^*\in \mathtt{Usef}^*(N_h)} |N_h\cup N_h^*\to \mathcal{F}||N_h^*\to \mathcal{H}_h^*| \leq \frac{4\prod_{i=2}^k d_i^{e_i(\mathcal{H}_h)-e_i(\mathcal{H}_h^-)-e_i(\mathcal{N}_h)}}{d_k^{\Delta^3}}|\mathcal{H}_h^-|_{\mathcal{G}}. \tag{8}$$

To bound the second sum in (6), we define $\mathcal{H}_h' := \mathcal{H}_h^* - \mathcal{N}_h^*$, and observe that trivially any copy N_h^* of $\mathcal{N}_h^*$ satisfies $|N_h^*\to\mathcal{H}_h^*|\leq |\mathcal{H}_h'|_{\mathcal{G}}$. Note that $\mathcal{H}_h'$ is nonempty. On the other hand, by the induction hypothesis applied several times,

$$\begin{aligned}|\mathcal{H}_h'|_{\mathcal{G}} &\leq ((1-\alpha)n)^{|\mathcal{H}_h'|-|\mathcal{H}_h^-|}\left(\prod_{i=2}^k d_i^{e_i(\mathcal{H}_h')-e_i(\mathcal{H}_h^-)}\right)|\mathcal{H}_h^-|_{\mathcal{G}}\\ &\leq \frac{2|\mathcal{H}_h^-|_{\mathcal{G}}}{\left(\prod_{i=2}^k d_i\right)^{2\Delta^4} n^{|\mathcal{H}_h^-|-|\mathcal{H}_h'|}}.\end{aligned}$$

Since at most $2\varepsilon_{k-1}^{1/4}|\mathcal{N}_h^*|_{\mathcal{G}}\leq 2\varepsilon_{k-1}^{1/4}n^{|\mathcal{N}_h^*|}$ copies of $\mathcal{N}_h^*$ do not lie in $\mathtt{Usef}(N_h)$, the second sum in (6) is bounded by

$$\begin{aligned}\sum_{N_h^*\notin \mathtt{Usef}(N_h)} &|N_h\cup N_h^*\to\mathcal{F}||N_h^*\to\mathcal{H}_h^*|\\ &\leq 2\varepsilon_{k-1}^{1/4}n^{|\mathcal{N}_h^*|}n^{|\mathcal{F}|-|\mathcal{N}_h|-|\mathcal{N}_h^*|}\frac{2|\mathcal{H}_h^-|_{\mathcal{G}}}{\left(\prod_{i=2}^k d_i\right)^{2\Delta^4}n^{|\mathcal{H}_h^-|-|\mathcal{H}_h'|}}\\ &= 2\varepsilon_{k-1}^{1/4}\frac{2|\mathcal{H}_h^-|_{\mathcal{G}}}{\left(\prod_{i=2}^k d_i\right)^{2\Delta^4}}\\ &\leq \left(\prod_{i=2}^k d_i^{e_i(\mathcal{H}_h)-e_i(\mathcal{H}_h^-)-e_i(\mathcal{N}_h)}\right)|\mathcal{H}_h^-|_{\mathcal{G}}. \qquad (9)\end{aligned}$$

The last inequality follows since $\varepsilon_{k-1}\ll d_2,d_3,\dots,d_k,1/\Delta$. Substituting (8) and (9) into (6) we obtain

$$\begin{aligned}|N_h\to\mathcal{H}_h| &\leq \left(1+\frac{4}{d_k^{\Delta^3}}\right)\left(\prod_{i=2}^k d_i^{e_i(\mathcal{H}_h)-e_i(\mathcal{H}_h^-)-e_i(\mathcal{N}_h)}\right)|\mathcal{H}_h^-|_{\mathcal{G}}\\ &\leq \frac{5\left(\prod_{i=2}^k d_i^{e_i(\mathcal{H}_h)-e_i(\mathcal{H}_h^-)-e_i(\mathcal{N}_h)}\right)}{d_k^{\Delta^3}}|\mathcal{H}_h^-|_{\mathcal{G}}. \qquad (10)\end{aligned}$$

It now remains only to relate $|\mathcal{H}_h^-|_{\mathcal{G}}$ to $|\mathcal{H}_h|_{\mathcal{G}}/|\mathcal{N}_h|_{\mathcal{G}}$. Once again we apply the induction hypothesis several times to obtain

$$|\mathcal{H}_h|_{\mathcal{G}} \geq ((1-\alpha)n)^{|\mathcal{H}_h|-|\mathcal{H}_h^-|} \prod_{i=2}^{k} d_i^{e_i(\mathcal{H}_h)-e_i(\mathcal{H}_h^-)} |\mathcal{H}_h^-|_{\mathcal{G}}.$$

On the other hand, the counting lemma implies that $|\mathcal{N}_h|_{\mathcal{G}} \leq (1+\alpha)\cdot \big(\prod_{i=2}^k d_i^{e_i(\mathcal{N}_h)}\big) n^{|\mathcal{N}_h|}$. Putting these two bounds together, we obtain

$$\begin{aligned} \frac{|\mathcal{H}_h|_{\mathcal{G}}}{|\mathcal{N}_h|_{\mathcal{G}}} &\geq \frac{((1-\alpha)n)^{|\mathcal{H}_h|-|\mathcal{H}_h^-|}\big(\prod_{i=2}^k d_i^{e_i(\mathcal{H}_h)-e_i(\mathcal{H}_h^-)}\big)|\mathcal{H}_h^-|}{(1+\alpha)\big(\prod_{i=2}^k d_i^{e_i(\mathcal{N}_h)}\big)n^{|\mathcal{N}_h|}} \\ &\geq \frac{1}{2}\left(\prod_{i=2}^{k} d_i^{e_i(\mathcal{H}_h)-e_i(\mathcal{H}_h^-)-e_i(\mathcal{N}_h)}\right)|\mathcal{H}_h^-|_{\mathcal{G}}. \end{aligned} \tag{11}$$

Together with (10), this shows that

$$|N_h \to \mathcal{H}_h| \leq \frac{5\cdot 2}{d_k^{\Delta^3}}\frac{|\mathcal{H}_h|_{\mathcal{G}}}{|\mathcal{N}_h|_{\mathcal{G}}},$$

which completes the proof of the claim. ∎

Using the claim we now go on to prove the induction step. Given a copy H_h of $\mathcal{H}_h$, we denote by $N_h(H_h)$ the induced copy of $\mathcal{N}_h$. We have

$$\begin{aligned} |\mathcal{H}|_{\mathcal{G}} &= \sum_{H_h\subseteq\mathcal{G}} |H_h \to \mathcal{H}| \geq \sum_{H_h\subseteq\mathcal{G}} (|N_h(H_h) \to \mathcal{B}| - cn) \\ &= \sum_{N_h\subseteq\mathcal{G}} |N_h \to \mathcal{H}_h||N_h \to \mathcal{B}| - cn|\mathcal{H}_h|_{\mathcal{G}} \\ &\geq (1-\varepsilon_k)\overline{|\mathcal{N}_h \to \mathcal{B}|}\Bigg(\sum_{N_h\subseteq\mathcal{G}} |N_h \to \mathcal{H}_h| - \sum_{N_h\notin\texttt{typ}} |N_h \to \mathcal{H}_h|\Bigg) - cn|\mathcal{H}_h|_{\mathcal{G}}. \end{aligned} \tag{12}$$

We want to show that the term in this expression which comes from the atypical copies of $\mathcal{N}_h$ does not affect the calculations too much, and so we aim to bound the contribution from atypical copies of $\mathcal{N}_h$. We have

$$\sum_{N_h\notin\texttt{typ}} |N_h \to \mathcal{H}_h| = \sum_{N_h\notin\texttt{typ}, N_h\in\texttt{Usef}} |N_h \to \mathcal{H}_h| + \sum_{N_h\notin\texttt{typ}, N_h\notin\texttt{Usef}} |N_h \to \mathcal{H}_h|. \tag{13}$$

Now the claim implies that we can bound the first sum in (13) by

$$\begin{aligned}\sum_{N_h \notin \mathtt{typ}, N_h \in \mathtt{Usef}} |N_h \to \mathcal{H}_h| &\leq \sum_{N_h \notin \mathtt{typ}, N_h \in \mathtt{Usef}} \frac{10}{d_k^{\Delta^3}} \frac{|\mathcal{H}_h|_{\mathcal{G}}}{|\mathcal{N}_h|_{\mathcal{G}}} \\ &\leq |\mathtt{atyp}| \frac{10}{d_k^{\Delta^3}} \frac{|\mathcal{H}_h|_{\mathcal{G}}}{|\mathcal{N}_h|_{\mathcal{G}}} \leq \sqrt{\varepsilon_k} |\mathcal{H}_h|_{\mathcal{G}}. \end{aligned} \tag{14}$$

Meanwhile we can also bound the second sum by

$$\begin{aligned}\sum_{N_h \notin \mathtt{typ}, N_h \notin \mathtt{Usef}} |N_h \to \mathcal{H}_h| \ &\leq \ \sum_{N_h \notin \mathtt{typ}, N_h \notin \mathtt{Usef}} |\mathcal{H}_h^-|_{\mathcal{G}} \\ &\overset{(11)}{\leq} (|\mathcal{N}_h|_{\mathcal{G}} - |\mathtt{Usef}|) \frac{2}{\prod_{i=1}^k d_i^{\Delta^2}} \frac{|\mathcal{H}_h|_{\mathcal{G}}}{|\mathcal{N}_h|_{\mathcal{G}}} \\ &\overset{(5)}{\leq} \varepsilon_{k-1}^{1/5} |\mathcal{H}_h|_{\mathcal{G}}. \end{aligned} \tag{15}$$

Combining (13), (14) and (15), we have

$$\sum_{N_h \notin \mathtt{typ}} |N_h \to \mathcal{H}_h| \leq 2\sqrt{\varepsilon_k} |\mathcal{H}_h|_{\mathcal{G}}$$

and combining this with (12), we obtain

$$\begin{aligned} |\mathcal{H}|_{\mathcal{G}} &\geq (1-\varepsilon_k)\overline{|\mathcal{N}_h \to \mathcal{B}|}\,(|\mathcal{H}_h|_{\mathcal{G}} - 2\sqrt{\varepsilon_k}|\mathcal{H}_h|_{\mathcal{G}}) - cn|\mathcal{H}_h|_{\mathcal{G}} \\ &= (1-\varepsilon_k) n \left(\prod_{i=2}^k d_i^{e_i(\mathcal{B}) - e_i(\mathcal{N}_h)} \right) (1 - 2\sqrt{\varepsilon_k}) |\mathcal{H}_h|_{\mathcal{G}} - cn |\mathcal{H}_h|_{\mathcal{G}} \\ &\geq (1-\alpha) n \left(\prod_{i=2}^k d_i^{e_i(\mathcal{H}) - e_i(\mathcal{H}_h)} \right) |\mathcal{H}_h|_{\mathcal{G}}, \end{aligned}$$

as required. This completes the proof of Theorem 3.

5. The regularity lemma for k-uniform hypergraphs

5.1. Preliminary definitions and statement

In this section we state the version of the regularity lemma for k-uniform hypergraphs due to Rödl and Schacht [27], which we use in the proof of Theorem 1 in the next section. To prepare for this we will first need some notation. We follow [27]. Given a finite set V of vertices, we will define a family $\mathcal{P} = \{\mathcal{P}^{(1)}, \dots, \mathcal{P}^{(k-1)}\}$ where each $\mathcal{P}^{(j)}$ is a partition of certain j-subsets of V. These partitions will satisfy properties which we will describe below. We denote by $[V]^j$ the set of all j-subsets of V. Suppose that we are given a partition $\mathcal{P}^{(1)} = \{V_1, \dots, V_{|\mathcal{P}^{(1)}|}\}$ of $[V]^1 = V$. We will call the V_i *clusters*. We denote by $\mathrm{Cross}_j = \mathrm{Cross}_j(\mathcal{P}^{(1)})$ the set of all those j-subsets of V that meet each part of $\mathcal{P}^{(1)}$ in at most 1 element. Each $\mathcal{P}^{(j)}$ will be a partition of Cross_j. Moreover, any two j-sets that belong to the same part of $\mathcal{P}^{(j)}$ will meet the same j clusters. This means that each part of $\mathcal{P}^{(j)}$ can be viewed as a j-partite j-uniform hypergraph whose vertex classes are these clusters. In particular, the parts of $\mathcal{P}^{(2)}$ can be thought of as bipartite subgraphs between two of the clusters. Moreover, for each part A of $\mathcal{P}^{(3)}$ there will be 3 clusters and 3 bipartite graphs belonging to $\mathcal{P}^{(2)}$ between these clusters such that all the 3-sets in A form triangles in the union of these 3 bipartite graphs.

More generally, suppose that we have already defined partitions $\mathcal{P}^{(1)}, \dots, \mathcal{P}^{(j-1)}$ and are about to define $\mathcal{P}^{(j)}$. Given $i < j$ and $I \in \mathrm{Cross}_i$, we let $P^{(i)}(I)$ denote the part of $\mathcal{P}^{(i)}$ the set I belongs to. Given $J \in \mathrm{Cross}_j$, the *polyad* $\hat{P}^{(j-1)}(J)$ *of* J is defined by

$$\hat{P}^{(j-1)}(J) := \bigcup \{P^{(j-1)}(I) : \ I \in [J]^{j-1}\}.$$

Thus $\hat{P}^{(j-1)}(J)$ is the unique collection of j parts of $\mathcal{P}^{(j-1)}$ in which J spans a copy of the complete $(j-1)$-uniform hypergraph $K_j^{(j-1)}$ on j vertices. Moreover, note that $\hat{P}^{(j-1)}(J)$ can be viewed as a j-partite $(j-1)$-uniform hypergraph whose vertex classes are the j clusters containing the vertices of J. We set

$$\hat{\mathcal{P}}^{(j-1)} := \{\hat{P}^{(j-1)}(J) : \ J \in \mathrm{Cross}_j\}.$$

Note that the polyads $\hat{P}^{(j-1)}(J)$ and $\hat{P}^{(j-1)}(J')$ need not be distinct for different $J, J' \in [V]^j$. However, if these polyads are distinct then $\mathcal{K}_j(\hat{P}^{(j-1)}(J)) \cap \mathcal{K}_j(\hat{P}^{(j-1)}(J')) = \emptyset$. (Recall that $\mathcal{K}_j(\hat{P}^{(j-1)}(J))$ is the set of all j-sets of vertices which form a $K_j^{(j-1)}$ in $\hat{P}^{(j-1)}(J)$. So in particular, $\mathcal{K}_j(\hat{P}^{(j-1)}(J))$ contains J.) This implies that $\{\mathcal{K}_j(\hat{P}^{(j-1)}) : \hat{P}^{(j-1)} \in \hat{\mathcal{P}}^{(j-1)}\}$ is a partition of Cross_j. The property of $\mathcal{P}^{(j)}$ which we require is that it refines $\{\mathcal{K}_j(\hat{P}^{(j-1)}) : \hat{P}^{(j-1)} \in \hat{\mathcal{P}}^{(j-1)}\}$, i.e., each part of $\mathcal{P}^{(j)}$ has to be contained in some $\mathcal{K}_j(\hat{P}^{(j-1)})$.

We also need a notion which generalizes that of a polyad: given $J \in \mathrm{Cross}_j$ and $i < j$ we set

$$\hat{P}^{(i)}(J) := \bigcup \{P^{(i)}(I) : \ I \in [J]^i\}.$$

Then the properties of our partitions imply that $\bigcup_{i=1}^{j-1} \hat{P}^{(i)}(J)$ is a $(j-1,j)$-complex.

Altogether, given $\boldsymbol{a} = (a_1,\dots,a_{k-1}) \in \mathbb{N}^{k-1}$ we say that $\mathcal{P}(k-1,\boldsymbol{a}) = \{\mathcal{P}^{(1)},\dots,\mathcal{P}^{(k-1)}\}$ is a *family of partitions on* V if

1. $\mathcal{P}^{(1)}$ is a partition of V into a_1 clusters.
2. For all $j = 2,\dots,k-1$, $\mathcal{P}^{(j)}$ is a partition of Cross_j such that for each part there is a polyad $\hat{P}^{(j-1)} \in \hat{\mathcal{P}}^{(j-1)}$ so that the part is contained in $\mathcal{K}_j(\hat{P}^{(j-1)})$. Moreover, for each polyad $\hat{P}^{(j-1)} \in \hat{\mathcal{P}}^{(j-1)}$, the set $\mathcal{K}_j(\hat{P}^{(j-1)})$ is the union of a_j parts of $\mathcal{P}^{(j)}$.

We say that $\mathcal{P}=\mathcal{P}(k-1,\boldsymbol{a})$ is t-*bounded* if $a_1,\dots,a_{k-1} \le t$. Suppose that a_1 divides $|V|$. Then $\mathcal{P}=\mathcal{P}(k-1,\boldsymbol{a})$ is called $(\eta,\delta,\boldsymbol{a})$-*equitable* if

1. $\mathcal{P}^{(1)}$ is a partition of V into a_1 clusters of equal size;
2. $|[V]^k \setminus \mathrm{Cross}_k| \le \eta\binom{|V|}{k}$;
3. for every $K \in \mathrm{Cross}_k$, the $(k-1,k)$-complex $\bigcup_{i=1}^{k-1} \hat{P}^{(i)}(K)$ is $(\mathbf{d},\delta,\delta,1)$-regular, where $\mathbf{d}=(1/a_{k-1},\dots,1/a_2)$.

In particular, the second condition implies that $1/a_1$ is small compared to η.

Let $\delta_k > 0$ and $r \in \mathbb{N}$. Suppose that $\mathcal{G}$ is a k-uniform hypergraph on V and $\mathcal{P}=\mathcal{P}(k-1,\boldsymbol{a})$ is a family of partitions on V. Recall that we can view each polyad $\hat{P}^{(k-1)} \in \hat{\mathcal{P}}^{(k-1)}$ as a $(k-1)$-uniform k-partite hypergraph. $\mathcal{G}$ is called (δ_k,r)-*regular with respect to* $\hat{P}^{(k-1)}$ if $\mathcal{G}$ is (d,δ_k,r)-regular with respect to $\hat{P}^{(k-1)}$ for some d. We say that $\mathcal{G}$ is (δ_k,r)-*regular with respect to* $\mathcal{P}$ if

$$\left|\bigcup\{\mathcal{K}_k(\hat{P}^{(k-1)}) : \mathcal{G} \text{ is not } (\delta_k,r)\text{-regular with respect to } \hat{P}^{(k-1)} \in \hat{\mathcal{P}}^{(k-1)}\}\right| \le \delta_k |V|^k.$$

This means that not much more than a δ_k-fraction of the k-subsets of V form a $K_k^{(k-1)}$ that lies within a polyad with respect to which $\mathcal{G}$ is not regular.

Now, we are ready to state the regularity lemma, which we are going to use in the proof of Theorem 1.

Theorem 9 **(Rödl and Schacht [27]).** *Let $k \ge 2$ be a fixed integer. For all positive constants η and δ_k and all functions $r : \mathbb{N}^{k-1} \to \mathbb{N}$ and $\delta : \mathbb{N}^{k-1} \to (0,1]$, there are integers t and m_0 such that the following holds for all $m \ge m_0$ which are divisible by $t!$. Suppose that $\mathcal{G}$ is a k-uniform hypergraph of order m. Then there exists an $\boldsymbol{a} \in \mathbb{N}^{k-1}$ and a family of partitions $\mathcal{P}=\mathcal{P}(k-1,\boldsymbol{a})$ of the vertex set V of $\mathcal{G}$ such that*

1. *$\mathcal{P}$ is $(\eta,\delta(\boldsymbol{a}),\boldsymbol{a})$-equitable and t-bounded; and*
2. *$\mathcal{G}$ is $(\delta_k,r(\boldsymbol{a}))$-regular with respect to $\mathcal{P}$.*

The advantage of this regularity lemma compared to the one proved earlier by Rödl and Skokan [29] is that it uses only two regularity constants δ and δ_k instead of $k-1$ different ones. The regularity constants $\delta_2,\dots,\delta_k$ produced by the regularity lemma in [29] might satisfy $\delta_2 \ll 1/a_2 \ll \delta_3 \ll 1/a_3 \ll \cdots \ll 1/a_{k-1} \ll \delta_k$, which would make the proof of the corresponding embedding theorem more technical in appearance.

Note that the constants in Theorem 9 can be chosen such that they satisfy the following hierarchy:

$$\frac{1}{m_0} \ll \frac{1}{r} = \frac{1}{r(\boldsymbol{a})}, \delta = \delta(\boldsymbol{a}) \ll \min\{\delta_k, \eta, 1/a_1, 1/a_2, \dots, 1/a_{k-1}\}. \tag{16}$$

5.2. The reduced hypergraph

In the proof of Theorem 1 that follows in the next section, we will use the so-called reduced hypergraph. If $\mathcal{P} = \{\mathcal{P}^{(1)},\dots,\mathcal{P}^{(k-1)}\}$ is the partition of the vertex set of $\mathcal{G}$ given by the regularity lemma, the *reduced hypergraph* $\mathcal{R} = \mathcal{R}(\mathcal{G},\mathcal{P})$ is a k-uniform hypergraph whose vertices are the clusters, i.e., the parts of $\mathcal{P}^{(1)}$. To define the set of hyperedges we need the following notion. We say that a k-tuple of clusters is *fruitful* if $\mathcal{G}$ is (δ_k, r)-regular with respect to all but at most a $\sqrt{\delta_k}$-fraction of all those polyads $\hat{P}^{(k-1)}$ which are induced on these k clusters. The set of hyperedges of $\mathcal{R}$ consists of precisely those k-tuples that are fruitful. In the proof of Theorem 1, we shall need an estimate on the number of these hyperedges. In particular, we need to show that $\mathcal{R}$ is very dense. This is conveyed in the following proposition.

Proposition 10. *All but at most $2\sqrt{\delta_k}a_1^k$ of the k-tuples of clusters are fruitful.*

Proof. By the dense counting lemma (Lemma 6) each polyad in $\hat{\mathcal{P}}^{(k-1)}$ contains at least

$$f(m,\boldsymbol{a}) := \frac{1}{2}\left(\frac{m}{a_1}\right)^k \prod_{i=2}^{k-1}\left(\frac{1}{a_i}\right)^{\binom{k}{i}}$$

copies of $K_k^{(k-1)}$. Since $\mathcal{G}$ is (δ_k, r)-regular with respect to $\mathcal{P}$, the number of polyads in $\hat{\mathcal{P}}^{(k-1)}$ with respect to which $\mathcal{G}$ is not (δ_k, r)-regular is at most

$$\frac{\delta_k m^k}{f(m,\boldsymbol{a})} = \frac{2\prod_{i=1}^{k-1} a_i^{\binom{k}{i}}}{m^k}\delta_k m^k = 2\delta_k \prod_{i=1}^{k-1} a_i^{\binom{k}{i}}. \tag{17}$$

We call these polyads *bad*. Now, each k-tuple of clusters induces $\prod_{i=2}^{k-1} a_i^{\binom{k}{i}}$ polyads in $\hat{\mathcal{P}}^{(k-1)}$. Thus if there were more than $2\sqrt{\delta_k}a_1^k$ k-tuples of clusters each inducing more than $\sqrt{\delta_k}\prod_{i=2}^{k-1} a_i^{\binom{k}{i}}$ bad polyads, the total number of bad polyads would exceed the bound given in (17), yielding a contradiction. ∎

6. Proof of Theorem 1

We now give a brief outline of the proof of Theorem 1: consider any red/blue colouring of the hyperedges of $K_m^{(k)}$, where $m = C|\mathcal{H}|$ and C is a large constant depending only on k and the maximum degree of $\mathcal{H}$. We apply the hypergraph regularity lemma to the red subhypergraph $\mathcal{G}_{red}$ to obtain a reduced hypergraph $\mathcal{R}$ which is very dense. Thus the following fact will show that $\mathcal{R}$ contains a copy of $K_\ell^{(k)}$ with $\ell := R(K_{k\Delta}^{(k)})$.

Fact 11. *For all $\ell, k \in \mathbb{N}$ with $\ell \geq k$, every k-uniform hypergraph $\mathcal{R}$ on $t \geq \ell$ vertices with $e(\mathcal{R}) > \left(1 - \binom{\ell}{k}^{-1}\right)\binom{t}{k}$ contains a copy of $K_\ell^{(k)}$.*

Proof. Let $\mathcal{R}$ be as in the statement of the fact. Assume for the sake of contradiction that $\mathcal{R}$ is $K_\ell^{(k)}$-free. Then for each ℓ-subset S of $V(\mathcal{R})$, we have $e(\mathcal{R}[S]) \leq \binom{\ell}{k} - 1$. But note that

$$e(\mathcal{R}) = \binom{t-k}{\ell-k}^{-1} \sum_{S \subset V(\mathcal{R}), |S| = \ell} e(\mathcal{R}[S]).$$

Thus $e(\mathcal{R}) \leq \binom{t-k}{\ell-k}^{-1}\binom{t}{\ell}\left(\binom{\ell}{k} - 1\right)$. Now the observation that $\binom{t-k}{\ell-k}^{-1}\binom{t}{\ell}\binom{\ell}{k} = \binom{t}{k}$ yields the required contradiction. ∎

The copy of $K_\ell^{(k)}$ in $\mathcal{R}$ involves ℓ clusters and for each k-tuple of them the red hypergraph $\mathcal{G}_{red}$ is regular with respect to almost all of the polyads induced on it. We will then show that we can find a $(k-1, \ell)$-complex $\mathcal{S}$ on these clusters such that for each $j = 2, \ldots, k-1$ the restriction of its underlying j-uniform hypergraph $\mathcal{S}_j$ to any $(j+1)$-tuple of clusters is a polyad. Moreover, $\mathcal{G}_{red}$ will be regular with respect to $\mathcal{S}_{k-1}$. By combining $E(\mathcal{G}_{red}) \cap \mathcal{K}_k(\mathcal{S}_{k-1})$ with $\mathcal{S}$, we will obtain a regular k-complex $\mathcal{S}_{red}$. Similarly we obtain a k-complex $\mathcal{S}_{blue}$ which also turns out to be regular. We then consider the following red/blue colouring of $K_\ell^{(k)}$. We colour a hyperedge red if $\mathcal{G}_{red}$ has density at least $1/2$ with respect to the corresponding polyad in $\mathcal{S}_{k-1}$ and blue otherwise. By the definition of ℓ, we can find a monochromatic $K_{k\Delta}^{(k)}$. If it is red, then we can apply the embedding lemma to $\mathcal{S}_{red}$ to find a red copy of $\mathcal{H}$. This can be done since $\Delta(\mathcal{H}) \le \Delta$ implies that the chromatic number of $\mathcal{H}$ is at most $(k-1)\Delta+1 \le k\Delta$. If our monochromatic copy of $K_{k\Delta}^{(k)}$ is blue, then we can apply the embedding theorem to $\mathcal{S}_{blue}$ and obtain a blue copy of $\mathcal{H}$.

Proof of Theorem 1. Given Δ and k, we choose C to be a sufficiently large constant. We will describe the bounds that C has to satisfy at the end of the proof. Let $m := C|\mathcal{H}|$ and consider any red/blue colouring of the hyperedges of $K_m^{(k)}$. Let $\mathcal{G}_{red}$ be the red and $\mathcal{G}_{blue}$ be the blue subhypergraph on $V = V(K_m^{(k)})$. We may assume without loss of generality that $e(\mathcal{G}_{red}) \ge e(\mathcal{G}_{blue})$. We apply the hypergraph regularity lemma to $\mathcal{G}_{red}$ with constants $\eta, \delta_k \ll 1/\Delta, 1/k$ as well as functions r and δ satisfying the hierarchy in (16). This gives us clusters $V_1, \dots, V_{a_1}$, each of size n say, together with a t-bounded $(\eta, \delta, \boldsymbol{a})$-equitable family of partitions $\mathcal{P} = \mathcal{P}(k-1, \boldsymbol{a})$ on V where $\boldsymbol{a} = (a_1, \dots, a_{k-1})$. (Note that by deleting some vertices of $\mathcal{G}_{red}$ if necessary we may assume that $m = |\mathcal{G}_{red}|$ is divisible by $t!$.) Since $\eta \ll 1/\Delta, 1/k$, condition (2) in the definition of an $(\eta, \delta, \boldsymbol{a})$-equitable family of partitions implies that the a_1 which we obtain from the regularity lemma satisfies

$$a_1 \ge R\big(K_{k\Delta}^{(k)}\big) =: \ell.$$

Note that the definition of ℓ involves a hypergraph Ramsey number whose value is unknown. However, for the argument below all we need is that this number exists.

Let $\mathcal{R}$ denote the reduced hypergraph, defined in the previous section. Proposition 10 implies that $\mathcal{R}$ has at least $(1-\varepsilon)\binom{a_1}{k}$ hyperedges, where $\varepsilon := 4\sqrt{\delta_k}k!$. Since $\delta_k \ll 1/\Delta, 1/k$, we may assume that $e(\mathcal{R}) \ge (1-\varepsilon)\binom{|\mathcal{R}|}{k} > \big(1 - \binom{\ell}{k}^{-1}\big)\binom{|\mathcal{R}|}{k}$. Since $|\mathcal{R}| = a_1 \ge \ell$, this means that we can apply Fact 11 to $\mathcal{R}$ to obtain a copy of $K_\ell^{(k)}$ in $\mathcal{R}$. Without loss of generality we may assume that the vertices of this copy are the clusters $V_1, \dots, V_\ell$.

As mentioned above, we now want to find a $(k-1, \ell)$-complex $\mathcal{S}$ on these clusters such that for each $j = 2, \dots, k-1$ its underlying j-uniform hypergraph $\mathcal{S}_j$ is a union of parts of $\mathcal{P}^{(j)}$ and $\mathcal{G}_{red}$ is regular with respect to $\mathcal{S}_{k-1}$. We construct $\mathcal{S}$ inductively starting from the lower levels. To begin with, for each pair V_i, V_j $(1 \le i < j \le \ell)$ independently, we choose with probability $1/a_2$ one of the parts of $\mathcal{P}^{(2)}$ induced on V_i, V_j. $\mathcal{S}_2$ will be the union of these parts. Now suppose that we have chosen $\mathcal{S}_{j-1}$ such that its restriction to any j-tuple of clusters forms a polyad (clearly this is the case for $\mathcal{S}_2$). Now, if $\hat{P}^{(j-1)}$ is such a polyad, we choose a part of $\mathcal{P}^{(j)}$ uniformly at random among the a_j parts of $\mathcal{P}^{(j)}$ that form $\mathcal{K}_j(\hat{P}^{(j-1)})$, independently for each j-tuple of clusters. We let $\mathcal{S}$ be the $(k-1,\ell)$-complex thus obtained.

We will show that there is some choice of $\mathcal{S}$ such that for every k-tuple among the clusters $V_1,\dots,V_\ell$ the hypergraph $\mathcal{G}_{red}$ is (δ_k,r)-regular with respect to the restriction of $\mathcal{S}_{k-1}$ to this k-tuple. Note that $\mathcal{S}_{k-1}$ restricted to any particular k-tuple of clusters is in fact a polyad selected uniformly at random among all polyads $\hat{P}^{(k-1)}$ induced by these k clusters. Therefore, since all the k-tuples of clusters are fruitful, the definition of a fruitful k-tuple implies that the probability that $\mathcal{G}_{red}$ has the necessary regularity is at least

$$1-\sqrt{\delta_k}\binom{\ell}{k}>\frac{1}{2}.$$

The final inequality holds since we may assume that δ_k is sufficiently small compared to $1/\ell$. This shows the existence of a $(k-1,\ell)$-complex $\mathcal{S}$ with the required properties. In what follows, $P_{\mathcal{S}}$ will always denote a $(k-1)$-uniform subhypergraph of $\mathcal{S}$ induced by k of the clusters $V_1,\dots,V_\ell$. So each such $P_{\mathcal{S}}$ is a polyad and to each hyperedge of the subhypergraph of $\mathcal{R}$ induced by the clusters $V_1,\dots,V_\ell$ there corresponds such a polyad $P_{\mathcal{S}}$.

We now use the densities of $\mathcal{G}_{red}$ with respect to $\mathcal{S}_{k-1}$ to define a red/blue colouring of the $K_\ell^{(k)}$ which we found in $\mathcal{R}$: we colour a hyperedge of this $K_\ell^{(k)}$ red if the polyad $P_{\mathcal{S}}$ corresponding to this hyperedge satisfies $d(\mathcal{G}_{red}|P_{\mathcal{S}})\geq 1/2$; otherwise we colour it blue. Since $\ell=R(K_{k\Delta}^{(k)})$, we find a monochromatic copy K of $K_{k\Delta}^{(k)}$ in our $K_\ell^{(k)}$. We now greedily assign the vertices of $\mathcal{H}$ to the clusters that form the vertex set of K in such a way that if k vertices of $\mathcal{H}$ form a hyperedge, then they are assigned to k different clusters. (We may think of this as a $(k\Delta)$-vertex-colouring of $\mathcal{H}$.) We now need to show that with this assignment we can apply the embedding lemma to find a monochromatic copy of $\mathcal{H}$ in either the subhypergraph of $\mathcal{G}_{red}$ induced by the $k\Delta$ clusters in K or the subhypergraph of $\mathcal{G}_{blue}$ induced by these clusters.

First suppose that K is red, so we want to apply the embedding theorem to the k-complex formed by $\mathcal{G}_{red}$ and $\mathcal{S}$ (induced on the $k\Delta$ clusters in K). However, the embedding theorem requires all the densities involved to be equal and of the from $1/a$ for $a\in\mathbb{N}$, whereas all we know is that for every polyad $P_{\mathcal{S}}$ corresponding to a hyperedge of K, we have $d(\mathcal{G}_{red}|P_{\mathcal{S}})\geq 1/2$. This minor obstacle can be overcome by choosing a subhypergraph $\mathcal{G}'_{red}\subseteq\mathcal{G}_{red}$ such that $\mathcal{G}'_{red}$ is $(1/2,3\delta_k,r)$-regular with respect to each polyad $P_{\mathcal{S}}$. The existence of such a $\mathcal{G}'_{red}$ follows immediately from the slicing lemma (Lemma 8). We then add $E(\mathcal{G}'_{red})\cap\mathcal{K}_k(\mathcal{S}_{k-1})$ to the subcomplex of $\mathcal{S}$ induced by the clusters in K to obtain a regular $(k,k\Delta)$-complex $\mathcal{S}_{red}$ and we apply the embedding theorem (Theorem 2) there to find a copy of $\mathcal{H}$ in $\mathcal{G}'_{red}$, and therefore also in $\mathcal{G}_{red}$.

On the other hand, if K is blue, we need to prove that $\mathcal{G}_{blue}$ is regular with respect to all chosen polyads $P_{\mathcal{S}}$. So suppose $\mathbf{Q}=(Q(1),\dots,Q(r))$ is an r-tuple of subhypergraphs of one of these polyads $P_{\mathcal{S}}$, satisfying $|\mathcal{K}_k(\mathbf{Q})|>\delta_k|\mathcal{K}_k(P_{\mathcal{S}})|$. Let d be such that $\mathcal{G}_{red}$ is (d,δ_k,r)-regular with respect to $P_{\mathcal{S}}$. Then

$$|(1-d)-d(\mathcal{G}_{blue}|\mathbf{Q})|=|d-(1-d(\mathcal{G}_{blue}|\mathbf{Q}))|=|d-d(\mathcal{G}_{red}|\mathbf{Q})|<\delta_k.$$

Thus $\mathcal{G}_{blue}$ is $(1-d,\delta_k,r)$-regular with respect to $P_{\mathcal{S}}$ (note that $\delta_k\ll 1/2\leq 1-d$). Following the same argument as in the previous case, we add $E(\mathcal{G}'_{blue})\cap\mathcal{K}_k(\mathcal{S}_{k-1})$ to the subcomplex of $\mathcal{S}$ induced by the clusters in K to derive the regular $(k,k\Delta)$-complex $\mathcal{S}_{blue}$ to which we can apply the embedding theorem to obtain a copy of $\mathcal{H}$ in $\mathcal{G}_{blue}$.

It remains to check that we can choose C to be a constant depending only on Δ and k. Note that the constants and functions η, δ_k, r and δ we defined at the beginning of the proof all depend only on Δ and k. So this is also true for the integers m_0 and t and the vector $\boldsymbol{a}=(a_1,\dots,a_{k-1})$ which we then obtained from the regularity lemma. Note that in order to be able to apply the regularity lemma to $\mathcal{G}_{red}$ we needed $m\geq m_0$, where $m=C|\mathcal{H}|$. This is certainly true if we set $C\geq m_0$. The embedding theorem allows us to embed subcomplexes of size at most cn, where n is the cluster size and where c satisfies $c\ll 1/a_2,\dots,1/a_{k-1},d_k,1/(k\Delta)$ (recall that $d_k=1/2$ and $d_i=1/a_i$ for all $i=2,\dots,k-1$). Thus c too depends only on Δ and k. In order to apply the embedding theorem we needed that $n\geq n_0$, where n_0 as defined in the embedding theorem depends only on Δ and k. Since the number of clusters is at most t, this is satisfied if $m\geq tn_0$, which in turn is certainly true if $C\geq tn_0$. When we applied the embedding lemma to $\mathcal{H}$, we needed that $|\mathcal{H}|\leq cn$. Since $n=m/a_1=C|\mathcal{H}|/a_1\geq C|\mathcal{H}|/t$, it suffices to choose $C\geq t/c$ for this. Altogether, this shows that we can define the constant C in Theorem 1 by $C:=\max\{m_0,tn_0,t/c\}$. ∎

7. Deriving Lemmas 4 and 6 from earlier work

First, we deduce Lemma 6 from [18, Cor. 6.11]. The difference between the two is that the latter result only counts complete hypergraphs but on the other hand it allows for different densities within each level. We need a few definitions that make this notion precise. Let $\mathcal{G}$ be a (k,t)-complex. Recall that $\mathcal{G}_i$ denotes the underlying i-uniform hypergraph of $\mathcal{G}$. For each $3\leq i<k$, we say that $\mathcal{G}_i$ *is* $(\geq d_i,\delta_i)$*-regular with respect to* $\mathcal{G}_{i-1}$, if for

every i-tuple Λ_i of vertex classes of $\mathcal{G}$ the induced hypergraph $\mathcal{G}_i[\Lambda_i]$ is (d_{Λ_i},δ_i)-regular with respect to $\mathcal{G}_{i-1}[\Lambda_i]$, for some $d_{\Lambda_i}\geq d_i$. Similarly we define when $\mathcal{G}_k$ *is* $(\geq d_k,\delta_k,r)$*-regular with respect to* $\mathcal{G}_{k-1}$ and when $\mathcal{G}_2$ *is* $(\geq d_2,\delta_2)$*-regular*. Let $\mathbf{d}:=(d_k,\dots,d_2)$. We say that a (k,t)-complex $\mathcal{G}$ is $(\geq \mathbf{d},\delta_k,\delta,r)$*-regular* if

- $\mathcal{G}_k$ is $(\geq d_k,\delta_k,r)$-regular with respect to $\mathcal{G}_{k-1}$;
- $\mathcal{G}_i$ is $(\geq d_i,\delta)$-regular with respect to $\mathcal{G}_{i-1}$ for each $3\leq i<k$;
- $\mathcal{G}_2$ is $(\geq d_2,\delta)$-regular.

Lemma 12 (Dense counting lemma for complete complexes [18]). *Let k,t,n_0 be positive integers and let $\varepsilon,d_2,\dots,d_{k-1},\delta$ be positive constants such that*

$$1/n_0 \ll \delta \ll \varepsilon \ll d_2,\dots,d_{k-1},1/t.$$

Then the following holds for all integers $n\geq n_0$. Suppose that $\mathcal{G}$ is a $(\geq(d_{k-1},\dots,d_2),\delta,\delta,1)$-regular $(k-1,t)$-complex with vertex classes $V_1,\dots,V_t$, all of size n. Then

$$\left|K_t^{(k-1)}\right|_{\mathcal{G}}=(1\pm\varepsilon)n^t\prod_{i=2}^{k-1}\prod_{\Lambda_i} d_{\Lambda_i},$$

where the second product is taken over all i-tuples Λ_i of vertex classes of $\mathcal{G}$.

We now show how to deduce Lemma 6 from this. Full details can be found in [5].

Proof of Lemma 6. First we prove the lemma for the case when $\ell=t$, i.e., when each of the vertex classes $X_1,\dots,X_t$ of $\mathcal{H}$ consists of exactly one vertex, say $X_i:=\{h_i\}$. Given such an $\mathcal{H}$ and a complex $\mathcal{G}$ as in Lemma 6, we construct a complex $\mathcal{G}'$ from $\mathcal{G}$ as follows: Starting with $i=2$, for all i with $2\leq i\leq k-1$ in turn, we successively consider each i-tuple $\Lambda_i=(V_{j_1},\dots,V_{j_i})$ of vertex classes of $\mathcal{G}$. If $h_{j_1},\dots,h_{j_i}$ forms an i-edge of $\mathcal{H}$ we let $\mathcal{G}'_i[\Lambda_i]=\mathcal{G}_i[\Lambda_i]$. If $h_{j_1},\dots,h_{j_i}$ does not form an i-edge we make each copy of $K_i^{(i-1)}$ in $\mathcal{G}'_{i-1}[\Lambda_i]$ into an i-edge of $\mathcal{G}'_i$. Thus in the latter case the density of $\mathcal{G}'_i[\Lambda_i]$ with respect to $\mathcal{G}'_{i-1}[\Lambda_i]$ will be 1. (If $i=2$, this means that we let $\mathcal{G}'_i[\Lambda_i]$ be the complete bipartite graph with vertex classes V_{j_1} and V_{j_2}.) Using that $\mathcal{H}$ is a complex, it is easy to see that $\mathcal{G}'$ is also $(\geq(d_{k-1},\dots,d_2),\delta,\delta,1)$-regular. Clearly, there is a bijection between the copies of $\mathcal{H}$ in $\mathcal{G}$ and the copies of $K_t^{(k-1)}$ in $\mathcal{G}'$. So $|\mathcal{H}|_{\mathcal{G}}=\left|K_t^{(k-1)}\right|_{\mathcal{G}'}$. The result now follows if we apply Lemma 12 to $\mathcal{G}'$.

It now remains to deduce Lemma 6 for arbitrary ℓ-partite complexes $\mathcal{H}$ from the result for the above case. For this, we use a simple argument that was also used in [6] to obtain Lemma 4 in the case $k=3$. We define a complex $\mathcal{G}^*$ from $\mathcal{G}$ by making $|X_i|$ copies $V_i^1,\dots,V_i^{|X_i|}$ of each vertex class V_i in such a way that for any selection of indices $i_1,\dots,i_t$ the complex $\mathcal{G}^*[V_1^{i_1},\dots,V_t^{i_t}]$ is isomorphic to $\mathcal{G}$. Note that $\mathcal{G}^*$ is $|\mathcal{H}|$-partite. Also, we can turn $\mathcal{H}$ into an $|\mathcal{H}|$-partite complex $\mathcal{H}^*$ by viewing each vertex as a single vertex class. Note that different copies of $\mathcal{H}$ in $\mathcal{G}$ give rise to different copies of $\mathcal{H}^*$ in $\mathcal{G}^*$. Thus $|\mathcal{H}|_{\mathcal{G}} \le |\mathcal{H}^*|_{\mathcal{G}^*}$. Conversely, the only case where a copy of $\mathcal{H}^*$ in $\mathcal{G}^*$ does not correspond to a copy of $\mathcal{H}$ in $\mathcal{G}$ is when there is some i and indices $j_1 \neq j_2$ such that the vertices that are used by $\mathcal{H}^*$ in $V_i^{j_1}$ and $V_i^{j_2}$ correspond to the same vertex of V_i. It is easy to see that the number of such copies is comparatively small. Thus the desired bounds on $|\mathcal{H}|_{\mathcal{G}}$ immediately follow from the bounds on $|\mathcal{H}^*|_{\mathcal{G}^*}$ which we obtained in the previous paragraph. ∎

We now prove Lemma 4. Its proof is based on the following version of the counting lemma that accompanies the hypergraph regularity lemma (Theorem 9) from [27]. Theorem 13 gives a lower bound on the number of complete complexes $K_t^{(k)}$ in a regular (k,t)-complex $\mathcal{G}$, under less restrictive assumptions on the regularity constants than those in Lemma 12.

Theorem 13 (Counting lemma for complete complexes [28]). *Let k,r,t,n_0 be positive integers and let $\varepsilon,d_2,\dots,d_k,\delta,\delta_k$ be positive constants such that $1/d_i\in\mathbb{N}$ for $i=2,\dots,k-1$ and*

$$1/n_0 \ll 1/r, \delta \ll \min\{\delta_k, d_2,\dots,d_{k-1}\} \le \delta_k \ll \varepsilon, d_k, 1/t.$$

Then the following holds for all integers $n \ge n_0$. Suppose that $\mathcal{G}$ is a $(\mathbf{d},\delta_k,\delta,r)$-regular (k,t)-complex with vertex classes $V_1,\dots,V_t$, all of size n, which respects the partition of $K_t^{(k)}$. Then

$$\big|K_t^{(k)}\big|_{\mathcal{G}} \ge (1-\varepsilon)n^t \prod_{i=2}^{k} d_i^{\binom{k}{i}}.$$

Lemma 4 is more general in the sense that it counts copies of complexes that may not be complete, and also gives an upper bound on their number. We will deduce Lemma 4 from Theorem 13 in several steps. The first (and main) step is to deduce a counting lemma which gives the number of copies of complete complexes, but now in a (k,t)-complex $\mathcal{G}$ where the density of $\mathcal{G}_i[\Lambda_i]$ with respect to $\mathcal{G}_{i-1}[\Lambda_i]$ might be different for different i-tuples Λ_i of vertex classes of $\mathcal{G}$.

Lemma 14 (Counting lemma for complete complexes – different densities). *Let k,r,t,n_0 be positive integers and let $\varepsilon,d_2,\dots,d_k,\delta,\delta_k$ be positive constants such that*

$$1/n_0 \ll 1/r, \delta \ll \min\{\delta_k, d_2,\dots,d_{k-1}\} \le \delta_k \ll \varepsilon, d_k, 1/t.$$

Then the following holds for all integers $n \geq n_0$*. Suppose* $\mathcal{G}$ *is a* $(\geq \mathbf{d}, \delta_k, \delta, r)$*-regular* (k,t)*-complex with vertex classes* $V_1, \dots, V_t$*, all of size* n*, such that for all* $2 < i < k$ *and all* i*-tuples* Λ_i *of vertex classes of* $\mathcal{G}$ *the hypergraph* $\mathcal{G}_i[\Lambda_i]$ *is* (d_{Λ_i}, δ)*-regular with respect to* $\mathcal{G}_{i-1}[\Lambda_i]$ *where* d_{Λ_i} *can be written as* $d_{\Lambda_i} = p_{\Lambda_i}/q_{\Lambda_i}$ *such that* $p_{\Lambda_i}, q_{\Lambda_i} \in \mathbb{N}$ *and* $1/q_{\Lambda_i} \geq d_i$*. Suppose that the analogue holds for all the* d_{Λ_2} *and all the* d_{Λ_k}*. Then*

$$\left|K_t^{(k)}\right|_{\mathcal{G}} = (1 \pm \varepsilon) n^t \prod_{i=2}^{k} \prod_{\Lambda_i} d_{\Lambda_i},$$

where the second product is taken over all i*-tuples* Λ_i *of vertex classes of* $\mathcal{G}$*.*

Proof. We will first prove the lower bound in this lemma by an inductive argument, in which we allow for different densities in the top levels but not in the lower levels, and show that we can always move down another level, until we allow different densities in all levels. This leads to the following definition. For any $2 < j \leq k$, we say that a complex $\mathcal{G}$ is $(\geq d_k, \dots, \geq d_j, d_{j-1}, \dots, d_2, \delta_k, \delta, r)$-*regular* if

- $\mathcal{G}_k$ is $(\geq d_k, \delta_k, r)$-regular with respect to $\mathcal{G}_{k-1}$;
- $\mathcal{G}_i$ is $(\geq d_i, \delta)$-regular with respect to $\mathcal{G}_{i-1}$ for each $j \leq i \leq k-1$;
- $\mathcal{G}_i$ is (d_i, δ)-regular with respect to $\mathcal{G}_{i-1}$ for each $3 \leq i \leq j-1$;
- $\mathcal{G}_2$ is (d_2, δ)-regular.

Choose new constants $\eta_i, \xi_i, \varepsilon_i$ and integers r_i satisfying

$$\begin{aligned} 1/n_0 \ll \delta = \xi_2 \ll \dots \ll \xi_k \ll \min\{\delta_k, d_2, \dots, d_{k-1}\} \\ \leq \delta_k = \eta_2 \ll \dots \ll \eta_{k+1} \ll \varepsilon_k \ll \dots \ll \varepsilon_2 = \varepsilon, d_k, 1/t \end{aligned}$$

and $1/n_0 \ll 1/r = 1/r_2 \ll \dots \ll 1/r_k \ll \min\{\delta_k, d_2, \dots, d_{k-1}\}$. Then the following claim immediately implies the lemma:

Claim. *Let* $2 \leq j \leq k$*. Suppose that* $\mathcal{G}$ *satisfies the conditions of Lemma 14 but is* $(\geq d_k, \dots, \geq d_j, d_{j-1}, \dots, d_2, \eta_j, \xi_j, r_j)$*-regular instead of* $(\geq \mathbf{d}, \delta_k, \delta, r)$*-regular if* $j > 2$*, where* $1/d_i \in \mathbb{N}$ *for all* $i = 2, \dots, j-1$*. Then*

$$\left|K_t^{(k)}\right|_{\mathcal{G}} \geq (1 - \varepsilon_j) n^t \left(\prod_{i=2}^{j-1} d_i^{\binom{t}{i}}\right) \prod_{i=j}^{k} \prod_{\Lambda_i} d_{\Lambda_i}.$$

We prove this claim by backward induction on j as follows: given a t-partite complex $\mathcal{G}$ which is $(\geq d_k, \dots, \geq d_j, d_{j-1}, \dots, d_2, \eta_j, \xi_j, r_j)$-regular, we will partition the hyperedges of $\mathcal{G}_j$ to obtain several $(\geq d_k, \dots, \geq d_{j+1}, d'_j, d_{j-1}, \dots, d_2, \eta_{j+1}, \xi_{j+1}, r_{j+1})$-regular complexes for some d'_j. We will then apply the lower bound from the induction hypothesis to each of these complexes. Summing over all of them will give the lower bound in the claim.

We first consider the case $j = k$. We will apply the slicing lemma (Lemma 8) to split the kth level $\mathcal{G}_k$ of the complex $\mathcal{G}$ to obtain regular complexes whose densities within the kth level are the same. Set $d'_k := 1/\prod_{\Lambda_k} q_{\Lambda_k}$. The slicing lemma implies that for all Λ_k there is a partition $P(\Lambda_k)$ of the set $E(\mathcal{G}_k[\Lambda_k])$ of k-edges induced on Λ_k such that each part is (d'_k, η_{k+1}, r_k)-regular with respect to $\mathcal{G}_{k-1}[\Lambda_k]$. So for each Λ_k, $P(\Lambda_k)$ has d_{Λ_k}/d'_k parts. Now for each Λ_k, choose one part from $P(\Lambda_k)$ and let $\mathcal{C}_k$ denote the resulting k-uniform t-partite hypergraph. Let $\mathcal{G}^{\mathcal{C}_k}$ denote the k-complex obtained from $\mathcal{G}$ by replacing $\mathcal{G}_k$ with $\mathcal{C}_k$. Then

$$\left|K_t^{(k)}\right|_{\mathcal{G}} = \sum_{\mathcal{C}_k} \left|K_t^{(k)}\right|_{\mathcal{G}^{\mathcal{C}_k}}.$$

Here the summation is over all possible choices of parts from each of the $\binom{t}{k}$ partitions $P(\Lambda_k)$. So the number of summands is $\prod_{\Lambda_k} d_{\Lambda_k}/d'_k = {d'}_k^{-\binom{t}{k}} \prod_{\Lambda_k} d_{\Lambda_k}$. Moreover, by Theorem 13 each summand in the above sum can be bounded below:

$$\left|K_t^{(k)}\right|_{\mathcal{G}^{\mathcal{C}_k}} \geq (1-\varepsilon_k) n^t \left(\prod_{i=2}^{k-1} d_i^{\binom{t}{i}} \right) {d'}_k^{\binom{t}{k}}.$$

Altogether, this implies the claim for $j = k$.

Now suppose that $j < k$ and that the claim holds for $j+1$. To apply the induction hypothesis, we now need to get equal densities in the jth level. We will achieve this by applying the slicing lemma (Lemma 8) to this level. Set $d'_j := 1/\prod_{\Lambda_j} q_{\Lambda_j}$. So $1/d'_j \in \mathbb{N}$. The slicing lemma implies that for every j-tuple Λ_j of vertex classes of $\mathcal{G}$ there is a partition $P(\Lambda_j)$ of the set $E(\mathcal{G}_j[\Lambda_j])$ of j-edges induced on Λ_j such that each part is (d'_j, ξ_{j+1})-regular with respect to $\mathcal{G}_{j-1}[\Lambda_j]$. For each Λ_j, the corresponding partition $P(\Lambda_j)$ will have $a_{\Lambda_j} := d_{\Lambda_j}/d'_j$ parts. Now for each Λ_j, choose one part from $P(\Lambda_j)$ and let $\mathcal{C}_j$ denote the resulting j-uniform t-partite hypergraph. We let $\mathcal{G}^{\mathcal{C}_j}$ denote the (k,t)-complex obtained from $\mathcal{G}$ as follows: we replace $\mathcal{G}_j$ by $\mathcal{C}_j$ and for each $j < i \leq k$ we replace $\mathcal{G}_i$ with the subhypergraph whose i-edges are all those i-sets of vertices that span a $K_i^{(j)}$ in $\mathcal{C}_j$. Thus $\mathcal{G}_j^{\mathcal{C}_j}$ is (d'_j, ξ_{j+1})-regular with respect to $\mathcal{G}_{j-1} = \mathcal{G}_{j-1}^{\mathcal{C}_j}$. However, to apply the induction hypothesis this is not enough. We also need to prove the following more general assertion.

(∗) *For all $i=j,\dots,k$ and any Λ_i the following holds. If $i=j$ then $\mathcal{G}_i^{\mathcal{C}_j}[\Lambda_i]$ is (d'_j,ξ_{j+1})-regular with respect to $\mathcal{G}_{i-1}^{\mathcal{C}_j}[\Lambda_i]$. If $j<i<k$ then $\mathcal{G}_i^{\mathcal{C}_j}[\Lambda_i]$ is $(d_{\Lambda_i},\xi_{j+1})$-regular with respect to $\mathcal{G}_{i-1}^{\mathcal{C}_j}[\Lambda_i]$. If $i=k$ then $\mathcal{G}_i^{\mathcal{C}_j}[\Lambda_i]$ is $(d_{\Lambda_i},\eta_{j+1},r_{j+1})$-regular with respect to $\mathcal{G}_{i-1}^{\mathcal{C}_j}[\Lambda_i]$ for all but at most $\sqrt{\eta_{j+1}}\prod_{\Lambda_j} a_{\Lambda_j}$ hypergraphs $\mathcal{C}_j$.*

We will prove (∗) by induction on i. If $i=j$ then we already know that the assertion is true. So suppose that $i>j$ and that the claim holds for $i-1$. We will first consider the case when $i<k$. The induction hypothesis together with the dense counting lemma for complete complexes (Lemma 12) implies that

$$|K_i^{(i-1)}|_{\mathcal{G}_{i-1}^{\mathcal{C}_j}[\Lambda_i]} \geq \frac{1}{2}n^i\left(\prod_{\ell=2}^{j-1} d_\ell^{\binom{i}{\ell}}\right) d'^{\binom{i}{j}}_j \prod_{s=j+1}^{i-1}\prod_{\Lambda_s\subseteq\Lambda_i} d_{\Lambda_s}. \tag{18}$$

Similarly, the assumptions on $\mathcal{G}$ in the claim together with Lemma 12 imply

$$|K_i^{(i-1)}|_{\mathcal{G}_{i-1}[\Lambda_i]} \leq 2n^i\left(\prod_{\ell=2}^{j-1} d_\ell^{\binom{i}{\ell}}\right)\prod_{s=j}^{i-1}\prod_{\Lambda_s\subseteq\Lambda_i} d_{\Lambda_s}. \tag{19}$$

If we combine these inequalities and use the fact that $\xi_j \ll \xi_{j+1} \ll d_j, 1/k$, we obtain

$$|K_i^{(i-1)}|_{\mathcal{G}_{i-1}^{\mathcal{C}_j}[\Lambda_i]} \geq \sqrt{\xi_{j+1}}|K_i^{(i-1)}|_{\mathcal{G}_{i-1}[\Lambda_i]} \geq \frac{\xi_j}{\xi_{j+1}}|K_i^{(i-1)}|_{\mathcal{G}_{i-1}[\Lambda_i]}. \tag{20}$$

In other words, a ξ_{j+1}-proportion of copies of $K_i^{(i-1)}$ in $\mathcal{G}_{i-1}^{\mathcal{C}_j}[\Lambda_i]$ gives rise to a ξ_j-proportion of copies in $\mathcal{G}_{i-1}[\Lambda_i]$. Moreover, $\mathcal{K}_i(\mathcal{G}_{i-1}^{\mathcal{C}_j}[\Lambda_i])\cap E(\mathcal{G}_i^{\mathcal{C}_j}[\Lambda_i]) = \mathcal{K}_i(\mathcal{G}_{i-1}^{\mathcal{C}_j}[\Lambda_i])\cap E(\mathcal{G}_i[\Lambda_i])$ by the definition of $\mathcal{G}^{\mathcal{C}_j}$ and so $d(\mathcal{G}_i^{\mathcal{C}_j}[\Lambda_i]|\mathcal{G}_{i-1}^{\mathcal{C}_j}[\Lambda_i]) = d(\mathcal{G}_i[\Lambda_i]|\mathcal{G}_{i-1}^{\mathcal{C}_j}[\Lambda_i]) = d_{\Lambda_i}\pm\xi_j$ by (20) and the (d_{Λ_i},ξ_j)-regularity of $\mathcal{G}_i[\Lambda_i]$ with respect to $\mathcal{G}_{i-1}[\Lambda_i]$. Thus the $(d_{\Lambda_i},\xi_{j+1})$-regularity of $\mathcal{G}_i^{\mathcal{C}_j}[\Lambda_i]$ with respect to $\mathcal{G}_{i-1}^{\mathcal{C}_j}[\Lambda_i]$ follows from the (d_{Λ_i},ξ_j)-regularity of $\mathcal{G}_i[\Lambda_i]$ with respect to $\mathcal{G}_{i-1}[\Lambda_i]$.

But if $i=k$, this might not be true, as η_{j+1} may not be small compared to d_j. However, given a k-tuple Λ_k of vertex classes of $\mathcal{G}$, it is true for most complexes $\mathcal{G}^{\mathcal{C}_j}[\Lambda_k]$. To see this, given Λ_k, let $\mathcal{B}$ be a (k,k)-complex obtained as follows: For each $\Lambda_j\subset\Lambda_k$, choose one part from $P(\Lambda_j)$ and let $\mathcal{B}_j$ denote the resulting j-uniform k-partite hypergraph. To obtain $\mathcal{B}$ from $\mathcal{G}[\Lambda_k]$, we replace $\mathcal{G}_j[\Lambda_k]$ by $\mathcal{B}_j$ and for each $j<i\leq k$ we replace $\mathcal{G}_i[\Lambda_k]$

with the subhypergraph whose i-edges are all those i-sets of vertices which span a $K_i^{(j)}$ in $\mathcal{B}_j$. Thus there are $\prod_{\Lambda_j \subset \Lambda_k} a_{\Lambda_j} =: A_{\Lambda_k}$ such complexes $\mathcal{B}$. (Recall that $a_{\Lambda_j} = d_{\Lambda_j}/d'_j$ was the number of parts of the partition $P(\Lambda_j)$.) Using that $(*)$ holds for all $i<k$, similarly as in (18)–(20) one can show that

$$(21)\qquad \left|K_k^{(k-1)}\right|_{\mathcal{B}_{k-1}} \geq \frac{{d'_j}^{\binom{k}{j}}}{4\prod_{\Lambda_j \subset \Lambda_k} d_{\Lambda_j}} \left|K_k^{(k-1)}\right|_{\mathcal{G}_{k-1}[\Lambda_k]} = \frac{\left|K_k^{(k-1)}\right|_{\mathcal{G}_{k-1}[\Lambda_k]}}{4A_{\Lambda_k}}.$$

We will now prove the following:

$(**)$ *The underlying k-uniform hypergraph $\mathcal{B}_k$ is not $(d_{\Lambda_k}, \eta_{j+1}, r_{j+1})$-regular with respect to $\mathcal{B}_{k-1}$ for less than $\eta_{j+1}A_{\Lambda_k}$ of the complexes $\mathcal{B}$.*

If $(**)$ is false then we can find $T := \eta_{j+1}A_{\Lambda_k}/2$ such complexes $\mathcal{B}^1, \dots, \mathcal{B}^T$, such that each $\mathcal{B}^\ell$ has a $\mathbf{Q}^\ell = (Q_1^\ell, \dots, Q_{r_{j+1}}^\ell)$ satisfying $Q_s^\ell \subseteq \mathcal{B}_{k-1}^\ell$ for all $s = 1, \dots, r_{j+1}$ and $\left|K_k^{(k-1)}\right|_{\mathbf{Q}^\ell} \geq \eta_{j+1}\left|K_k^{(k-1)}\right|_{\mathcal{B}_{k-1}^\ell}$, but either $d(\mathcal{B}_k^\ell|\mathbf{Q}^\ell) > d_{\Lambda_k} + \eta_{j+1}$ for each ℓ or $d(\mathcal{B}_k^\ell|\mathbf{Q}^\ell) < d_{\Lambda_k} - \eta_{j+1}$ for each ℓ. We will assume the latter – the proof in the former case is similar. But then let $\mathbf{Q} = (\mathbf{Q}^1, \mathbf{Q}^2, \dots, \mathbf{Q}^T)$. Thus $\mathbf{Q}$ is a Tr_{j+1}-tuple and

$$\left|K_k^{(k-1)}\right|_{\mathbf{Q}} \geq \sum_{\ell=1}^{T} \eta_{j+1}\left|K_k^{(k-1)}\right|_{\mathcal{B}_{k-1}^\ell} \stackrel{(21)}{\geq} \eta_j\left|K_k^{(k-1)}\right|_{\mathcal{G}_{k-1}[\Lambda_k]}.$$

Since we may assume that $Tr_{j+1} \leq r_j$ our assumption on the regularity of $\mathcal{G}_k[\Lambda_k]$ with respect to $\mathcal{G}_{k-1}[\Lambda_k]$ implies that $d(\mathcal{G}_k[\Lambda_k]|\mathbf{Q}) \geq d_{\Lambda_k} - \eta_j$. On the other hand, the definition of $\mathcal{B}$ implies that $d(\mathcal{B}_k^\ell|\mathbf{Q}^\ell) = d(\mathcal{G}_k[\Lambda_k]|\mathbf{Q}^\ell)$. Thus $d(\mathcal{G}_k[\Lambda_k]|\mathbf{Q}) \leq \max_{1\leq \ell \leq T} d(\mathcal{G}_k[\Lambda_k]|\mathbf{Q}^\ell) = \max_{1\leq \ell \leq T} d(\mathcal{B}_k^\ell|\mathbf{Q}^\ell) < d_{\Lambda_k} - \eta_{j+1}$. This is a contradiction, and so $(**)$ holds.

Note that $(**)$ implies that for all but at most $\binom{t}{k}\eta_{j+1}\prod_{\Lambda_j} a_{\Lambda_j}$ hypergraphs $\mathcal{C}_j$ the hypergraph $\mathcal{G}_k^{\mathcal{C}_j}$ is $(d_{\Lambda_k}, \eta_{j+1}, r_{j+1})$-regular with respect to $\mathcal{G}_{k-1}^{\mathcal{C}_j}$ – we call these $\mathcal{C}_j$ *nice*. Since $\eta_{j+1} \ll 1/t$, this completes the proof of $(*)$.

We are now ready to finish the proof of the induction step of the claim. The induction

$$\left|K_t^{(k)}\right|_{\mathcal{G}} \geq \sum_{\text{nice } \mathcal{C}_j} \left|K_t^{(k)}\right|_{\mathcal{G}^{\mathcal{C}_j}} \geq (1-\varepsilon_{j+1}) \sum_{\text{nice } \mathcal{C}_j} n^t \left(\prod_{i=2}^{j-1} d_i^{\binom{t}{i}}\right) {d'_j}^{\binom{t}{j}} \prod_{i=j+1}^{k} \prod_{\Lambda_i} d_{\Lambda_i}.$$

The summation is over all possible choices of nice $\mathcal{C}_j$. So the number of summands is at least $(1-\sqrt{\eta_{j+1}})\prod_{\Lambda_j} a_{\Lambda_j}$ and for each Λ_j we have $a_{\Lambda_j}d'_j =$

d_{Λ_j}. Since $\eta_{j+1}, \varepsilon_{j+1} \ll \varepsilon_j$, the claim follows and hence the lower bound in Lemma 14 as well.

It is straightforward to obtain a corresponding upper bound from the lower bound. The proof is based on an argument that was used in [24] and later in [6] to derive a similar upper bound in the case of 3-complexes and thus we only give a sketch of it. A detailed proof can be found in [5]. Let $[t]^k$ denote the set of all k-subsets of $[t] = \{1, \ldots, t\}$. Given $S \subseteq [t]^k$, we let $\mathcal{G}^S$ denote the (k,t)-complex obtained from $\mathcal{G}$ as follows: for each $\{i_1, \ldots, i_k\} \in S$ we replace the set $E_k(\mathcal{G}[\Lambda_k])$ of all k-edges of $\mathcal{G}$ induced on $\Lambda_k := \{V_{i_1}, \ldots, V_{i_k}\}$ by $\mathcal{K}_k(\mathcal{G}_{k-1}[\Lambda_k]) \setminus E_k(\mathcal{G}[\Lambda_k])$. Thus the density of $\mathcal{G}_k^S[\Lambda_k]$ with respect to $\mathcal{G}_{k-1}^S[\Lambda_k]$ is now $1 - d_{\Lambda_k}$. Moreover,

$$\left|K_t^{(k-1)}\right|_{\mathcal{G}_{k-1}} = \sum_{S \subseteq [t]^k} \left|K_t^{(k)}\right|_{\mathcal{G}^S}.$$

Observe that $\left|K_t^{(k)}\right|_{\mathcal{G}} = \left|K_t^{(k)}\right|_{\mathcal{G}^\emptyset}$ and hence

$$\left|K_t^{(k)}\right|_{\mathcal{G}} = \left|K_t^{(k-1)}\right|_{\mathcal{G}_{k-1}} - \sum_{S \subseteq [t]^k, S \neq \emptyset} \left|K_t^{(k)}\right|_{\mathcal{G}^S}.$$

Thus, to obtain an upper bound on $\left|K_t^{(k)}\right|_{\mathcal{G}}$ all we have to do now is to obtain an upper bound on $\left|K_t^{(k-1)}\right|_{\mathcal{G}_{k-1}}$ and a lower bound on $\left|K_t^{(k)}\right|_{\mathcal{G}^S}$, for every non-empty S. But the former follows from the dense counting lemma for complete complexes (Lemma 12) and the latter follows from the lower bound in Lemma 14, which we proved above. (This is why we need to allow more general densities than just $1/a$, for $a \in \mathbb{N}$.) ∎

Lemma 4 now follows from Lemma 14 in exactly the same way as Lemma 6 followed from Lemma 12.

8. Proof of the Extension Lemmas 5 and 7

We now use Lemma 4 to derive Lemma 5 (Lemma 7 can be derived in the same way from Lemma 6). The proof idea is similar to that of [28, Cor. 14], [10, Lemma 6.6] and [6, Lemma 5]. Pick a copy H of $\mathcal{H}$ in $\mathcal{G}$ uniformly at random, and define $X := |H \to \mathcal{H}'|$. Then X is a random variable. We have $\mathbb{E}(X) = \frac{1}{|\mathcal{H}|_{\mathcal{G}}} \sum_{H \in \mathcal{G}} |H \to \mathcal{H}'| = |\mathcal{H}'|_{\mathcal{G}} / |\mathcal{H}|_{\mathcal{G}}$. (Here the sum $\sum_{H \in \mathcal{G}}$ is over all copies of $\mathcal{H}$ in $\mathcal{G}$.) We pick some constant ε satisfying $\delta_k \ll \varepsilon \ll \beta$. By applying the upper bound of the counting lemma (Lemma 4) to $\mathcal{H}$ and the lower bound to $\mathcal{H}'$ we obtain a lower bound for $\mathbb{E}(X)$. Similarly we obtain an upper bound. In this way we can easily deduce that

$$\mathbb{E}(X) = (1 \pm \sqrt{\varepsilon})\overline{|\mathcal{H} \to \mathcal{H}'|}. \tag{22}$$

Now consider $\mathbb{E}(X^2)$. We aim to show that its value is approximately $\overline{|\mathcal{H}\to\mathcal{H}'|}^2$, and so X has a low variance. Using Chebyshev's inequality, this will then imply that X is concentrated around its mean. In other words, only a few copies of $\mathcal{H}$ do not extend to the correct number of copies of $\mathcal{H}'$ in $\mathcal{G}$.

Observe that $\mathbb{E}(X^2) = \frac{1}{|\mathcal{H}|_{\mathcal{G}}} \sum_{H\in\mathcal{G}} |H \to \mathcal{H}'|^2$. We view $|H \to \mathcal{H}'|^2$ as the number of pairs H_1', H_2' of copies of $\mathcal{H}'$ which extend H. Here the pairs are allowed to overlap, but we first obtain a rough estimate by insisting that they intersect precisely in H. So let $\mathcal{H}^*$ be the (k,ℓ)-complex obtained from two disjoint copies of $\mathcal{H}'$ by identifying them on $\mathcal{H}$. Thus any copy of $\mathcal{H}^*$ in $\mathcal{G}$ extending H corresponds to a pair H_1', H_2'. However, we will later need to take account of those pairs H_1', H_2' which do not arise from a copy of $\mathcal{H}^*$. These pairs are exactly those whose intersection is strictly larger than H.

By applying the counting lemma (Lemma 4) to $\mathcal{H}^*$ and to $\mathcal{H}$, as before we obtain

$$\frac{1}{|\mathcal{H}|_{\mathcal{G}}} \sum_{H\in\mathcal{G}} |H \to \mathcal{H}^*| = (1 \pm \sqrt{\varepsilon})\overline{|\mathcal{H} \to \mathcal{H}^*|} = (1 \pm \sqrt{\varepsilon})\overline{|\mathcal{H} \to \mathcal{H}'|}^2.$$

On the other hand, the number of pairs H_1', H_2' which do not arise from a copy of $\mathcal{H}^*$ is at most $(t'-t)^2 n^{2(t'-t)-1} < \varepsilon\big(\big(\prod_{i=2}^k d_i^{e_i(\mathcal{H}')-e_i(\mathcal{H})}\big)n^{t'-t}\big)^2 = \varepsilon\overline{|\mathcal{H}\to\mathcal{H}'|}^2$. Thus

$$\frac{1}{|\mathcal{H}|_{\mathcal{G}}} \sum_{H\in\mathcal{G}} |H \to \mathcal{H}'|^2 = (1 \pm 2\sqrt{\varepsilon})\overline{|\mathcal{H} \to \mathcal{H}'|}^2. \tag{23}$$

Putting (22) and (23) together, we obtain

$$\mathrm{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 < 5\sqrt{\varepsilon}\ \overline{|\mathcal{H} \to \mathcal{H}'|}^2.$$

Now recall Chebyshev's inequality: $\mathbb{P}(|X - \mathbb{E}(X)| \geq t) \leq var(X)/t^2$. We apply this inequality with $t := \beta\overline{|\mathcal{H}\to\mathcal{H}'|}$. This implies that the probability that a randomly chosen copy of $\mathcal{H}$ in $\mathcal{G}$ does not satisfy the conclusion of the extension lemma is at most $var(X)/\beta^2\overline{|\mathcal{H}\to\mathcal{H}'|}^2 < 5\sqrt{\varepsilon}/\beta^2 < \beta$, and so at most $\beta|\mathcal{H}|_{\mathcal{G}}$ copies of $\mathcal{H}$ do not satisfy the conclusion, as required.

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